Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations*

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Abstract

In this paper, we consider a global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations (ANS). In order to do so, we first introduce the scaling invariant Besov-Sobolev type spaces, $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ and $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$, $p \geq 2$. Then, we prove the global wellposedness for (ANS) provided the initial data are sufficient small compared to the horizontal viscosity in some suitable sense, which is stronger than $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ norm. In particular, our results imply the global wellposedness of (ANS) with high oscillatory initial data.

1 Introduction

1.1 Introduction to the anisotropic Navier-Stokes equations

In this paper, we are going to study the 3-D incompressible anisotropic Navier-Stokes equations (ANS), namely,

$$\begin{cases} u_t + u \cdot \nabla u - \nu_h \Delta_h u - \nu_3 \partial_{x_3}^2 u = -\nabla P, \\ \mathrm{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u(t,x)$ and $P(t,x)$ denote the fluid velocity and the pressure, respectively, the viscosity coefficients ν_h and ν_3 are two constants satisfying

$$\nu_h > 0, \ \nu_3 \geq 0,$$

$x = (x_h, x_3) \in \mathbb{R}^3$ and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$. When $\nu_h = \nu_3 = \nu$, such system is the classical (isotropic) Navier-Stokes system (NS). It is appeared in geophysical fluids (see for instance [4]). In fact, instead of putting the classical viscosity $-\nu\Delta$ in (NS), meteorologists often simulate the turbulent diffusion by putting a viscosity of the form $-\nu_h \Delta_h - \nu_3 \partial_{x_3}^2$, where ν_h and ν_3 are empiric constants, and ν_3 usually is much smaller than ν_h. We refer to the book of J. Pedlosky [14], Chapter 4 for a more complete discussion. In particular, in the studying of Ekman boundary layers for rotating fluids [4, 6, 8], it makes sense to consider anisotropic viscosities of the type $-\nu_h \Delta_h - \varepsilon\beta\partial_{x_3}^2$, where ε is a very small parameter. The system (ANS) has been studied first by J.Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [5] and D. Iftimie in [9], where the

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authors proved that such system is locally wellposed for initial data in the anisotropic Sobolev space

$$H^{0,\frac{1}{2}+\varepsilon} = \left\{ u \in L^2(\mathbb{R}^3); \|u\|^2_{\dot{H}^{\frac{1}{2}+\varepsilon}} = \int_{\mathbb{R}^3} |\xi_3|^{1+2\varepsilon} |\hat{u}(\xi_h, \xi_3)|^2 d\xi < +\infty \right\},$$

for some $\varepsilon > 0$. Moreover, it has also been proved that if the initial data u_0 is small enough in the sense of that

$$\|u_0\|^\varepsilon_{L^2} \|u_0\|^{1-\varepsilon}_{\dot{H}^{0,\frac{1}{2}+\varepsilon}} \leq c\nu_h \tag{1.2}$$

for some sufficiently small constant c, then the system (1.1) is global wellposed.

Similar to the classical Navier-Stokes equations, the system (ANS) has a scaling invariance. Indeed, if u is a solution of (ANS) on a time interval $[0, T]$ with initial data u_0, then the vector field u_λ defined by

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$$

is also a solution of (ANS) on the time interval $[0, \lambda^{-2}T]$ with the initial data $\lambda u_0(\lambda x)$. The smallness condition (1.2) is of course scaling invariant, but the norm $\|\cdot\|_{\dot{H}^{\frac{1}{2}+\varepsilon}}$ is not. M. Paicu proved in [12] a similar result for the system (ANS) with $\nu_3 = 0$ in the case of the initial data $u_0 \in B^{0,\frac{1}{2}}$. This space has a scaling invariant norm. Then J.Y. Chemin and P. Zhang [3] obtained a similar result in the scaling invariant space $B_4^{-\frac{1}{2},\frac{1}{2}}$. Considering the periodic anisotropic Naiver-Stokes equations, Paicu obtained the global wellposedness in [13].

On the other hand, the classical (isotropic) Navier-Stokes system (NS) is globally wellposed for small initial data in Besov norms of negative index. In [1], M. Cannone, Y. Meyer and F. Planchon proved that: if the initial data satisfy

$$\|u_0\|_{\dot{B}_{p,\infty}^{-1+\frac{3}{p}}} \leq c\nu,$$

for $p > 3$ and some constant c small enough, then the classical Navier-Stokes system (NS) is globally wellposed. Then, H. Koch and D. Tataru generalized this theorem to the BMO^{-1} norm (see [11]), D. Iftimie in [10] obtained the global wellposedness in anisotropic spaces H^{s_1,s_2,s_3} and $B^{0,\frac{1}{2}}$. Recently, J.Y. Chemin and I. Gallagher [2] proved that if a certain nonlinear function of the initial data is small enough, then there is a global solution to the Navier-Stokes equations (NS).

Let $\phi_0(x_3)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ satisfying $\text{supp}\hat{\phi}_0 \subset \mathcal{C}_v$, $\phi_1(x_h)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ satisfying $\text{supp}\hat{\phi}_1 \subset \mathcal{C}_h$, where $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) is a ring of $\mathbb{R}_h^2$ (resp. $\mathbb{R}_v$). The mentioned results imply that the system (NS) is globally wellposed for the initial data u_0^ε defined by

$$u_0^\varepsilon(x) = \varepsilon^{-\frac{1}{2}} \sin(\frac{x_1}{\varepsilon})(0, -\partial_{x_3}(\phi_0\phi_1), \partial_{x_2}(\phi_0\phi_1)) \tag{1.3}$$

with small enough ε. The goal of our work is to prove a result of this type for the anisotropic Navier-Stokes system (1.1).

1.2 Statement of the results.

As in [3], let us begin with the definition of the spaces, which we will be going to work with. It requires an anisotropic version of dyadic decomposition of the Fourier space, let us first recall the following operators of localization in Fourier space, for $(k, l) \in \mathbb{Z}^2$,

$$\Delta_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat{a}), \ \Delta_l^v a = \mathcal{F}^{-1}(\varphi(2^{-l}|\xi_3|)\hat{a}),$$

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$$S_k^h a = \sum_{k' \le k-1} \Delta_{k'}^h a, \ S_l^v a = \sum_{l' \le l-1} \Delta_{l'}^v a,$$

where $\mathcal{F}a$ or $\hat{a}$ denotes the Fourier transform of the function a, and φ is a function in $\mathcal{D}((\frac{3}{4}, \frac{8}{3}))$ satisfying

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1, \ \forall \ \tau > 0.$$

Our main motivation to introduce the following spaces is to find a scaling invariant Besov-Sobolve type space such that u_0^ε can be small. According to the definitions of $B^{0,\frac{1}{2}}$ (in [10, 12]) and $B_4^{-\frac{1}{2},\frac{1}{2}}$ (in [3]), we give the definition of $B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \in [2, \infty)$, as follows.

Definition 1.1. We denote by $B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \in [2, \infty)$, the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^3)$ by the following norm:

$$\|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} = \sum_{l \in \mathbb{Z}} 2^{\frac{l}{2}} \left(\sum_{k=l-1}^{\infty} 2^{(-2+\frac{4}{p})k} \|\Delta_k^h \Delta_l^v a\|_{L_h^p(L_v^2)}^2 \right)^{\frac{1}{2}} + \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|S_{j-1}^h \Delta_j^v a\|_{L^2(\mathbb{R}^3)}.$$

Let $B^{0,\frac{1}{2}} := B_2^{0,\frac{1}{2}}$ and $\|a\|_{B^{0,\frac{1}{2}}} \simeq \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v a\|_{L^2(\mathbb{R}^3)}$.

To study the evolution of (1.1) with initial data in $B_p^{-1+\frac{2}{p},\frac{1}{2}}$, we need also to introduce the following space.

Definition 1.2. We denote by $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ the space of distributions, which is the completion of $C^\infty([0, T]; \mathcal{S}(\mathbb{R}^3))$ by the following norm:

$$\begin{aligned}
&\|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)} \\
&= \sum_{l \in \mathbb{Z}} 2^{\frac{l}{2}} \left[\sum_{k=l-1}^{\infty} \left(2^{(-2+\frac{4}{p})k} \|\Delta_k^h \Delta_l^v a\|_{L_T^\infty(L_h^p(L_v^2))}^2 + \nu_h 2^{\frac{4}{p}k} \|\Delta_k^h \Delta_l^v a\|_{L_T^2(L_h^p(L_v^2))}^2 \right. \right. \\
&\quad \left. \left. + \nu_3 2^{(-2+\frac{4}{p})k+2l} \|\Delta_k^h \Delta_l^v a\|_{L_T^2(L_h^p(L_v^2))}^2 \right) \right]^{\frac{1}{2}} + \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left(\|S_{j-1}^h \Delta_j^v a\|_{L_T^\infty(L^2(\mathbb{R}^3))} \right. \\
&\quad \left. + \nu_h^{\frac{1}{2}} \|\nabla_h S_{j-1}^h \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} + \nu_3^{\frac{1}{2}} \|\partial_3 S_{j-1}^h \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} \right).
\end{aligned}$$

Let $B^{0,\frac{1}{2}}(T) := B_2^{0,\frac{1}{2}}(T)$ and

$$\|a\|_{B^{0,\frac{1}{2}}(T)} \simeq \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left(\|\Delta_j^v a\|_{L_T^\infty(L^2(\mathbb{R}^3))} + \nu_h^{\frac{1}{2}} \|\nabla_h \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} + \nu_3^{\frac{1}{2}} \|\partial_3 \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} \right).$$

In our global result, we need that the initial data u_0 and a certain nonlinear function of the initial data $u_F \cdot \nabla u_F$ are small enough in some suitable sense, where

$$u_F := e^{\nu_h t \Delta_h + \nu_3 t \partial_3^2} u_{0hh} \quad \text{and} \quad u_{0hh} = \sum_{k \ge l-1} \Delta_k^h \Delta_l^v u_0. \tag{1.4}$$

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Definition 1.3. We define $[\cdot]_{E_T^p}$ by:

$$[a]_{E_T^p} := \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} + \|a_F \cdot \nabla a_F\|_{L_T^1(B^{0,\frac{1}{2}})},$$

where

$$\|f\|_{L_T^1(B^{0,\frac{1}{2}})} := \int_0^T \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v f\|_{L^2(\mathbb{R}^3)} dt,$$

$$a_F := e^{\nu_h t\Delta_h + \nu_3 t\partial_3^2} a_{hh}, \quad a_{hh} := \sum_{k\geq l-1} \Delta_k^h \Delta_l^v a.$$

Now, we present the main results of this paper, which cover the results in [3, 12] and partial result in [10].

Theorem 1.1. *A constant c exists such that, if the divergence free vector field $u_0 \in B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \geq 2$ and $[u_0]_{E_\infty^p} \leq c\nu_h$, then, the system (1.1) with initial data u_0 has a unique global solution $u \in B_p^{-1+\frac{2}{p},\frac{1}{2}}(\infty) \cap C([0,\infty); B_p^{-1+\frac{2}{p},\frac{1}{2}})$, and $\|u\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(\infty)}$ is independent of ν_3.*

Furthermore, let $u_i \in B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ be the solution for the system (1.1) with initial data u_{0i}, $p \geq 2$, $i = 1, 2$. If $\nu_3 > 0$ and $u_{01} - u_{02} \in L^2$, then we have

$$\|u_1 - u_2\|_{L_T^\infty(L^2(\mathbb{R}^3))} \tag{1.5}$$

$$\leq \|u_{01} - u_{02}\|_{L^2(\mathbb{R}^3)} \exp\left\{ C\nu_h^{-1}(\nu_h^{-\frac{p+1}{p-1}} + \nu_3^{-\frac{p+1}{p-1}}) \left(\sum_{i=1}^2 \|u_i\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)} \right)^{\frac{2p}{p-1}} \right\}.$$

In what follows, we always use C to denote a generic positive constant independent of ν_3. Repeating the proof of Theorem 1.1, we may conclude the following theorem concerning local wellposedness for large data.

Theorem 1.2. *If the divergence free vector field $u_0 \in B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \geq 2$ and $[u_0]_{E_T^p} < \infty$, then there exists a constant $T_0 \in (0, T]$ such that the system (1.1) has a unique solution $u \in B_p^{-1+\frac{2}{p},\frac{1}{2}}(T_0) \cap C([0, T_0]; B_p^{-1+\frac{2}{p},\frac{1}{2}})$, and $\|u\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T_0)}$ is independent of ν_3.*

Remark 1.1. These theorems imply that the third viscosity coefficient ν_3 do not play a role except the continuous dependence (1.5).

Proposition 1.1. *If $p \in [2, 4]$, we have*

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \int_0^\infty \|\Delta_j^v(a_F \cdot \nabla a_F)\|_{L^2(\mathbb{R}^3)} dt \lesssim \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2$$

and

$$[a]_{E_\infty^p} \lesssim \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} + \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2.$$

Remark 1.2. This proposition will be proved in Section 6. It implies that if $p \in [2, 4]$, then the condition $[u_0]_{E_\infty^p} \leq c\nu_h$ in Theorem 1.1 can be replaced by

$$\|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} \leq c\nu_h,$$

and the condition $[u_0]_{E_T^p} < \infty$ in Theorem 1.2 can be omitted.

Remark 1.3. Using the similar argument to that in the proof of Proposition 1.1, we obtain

$$2^{\frac{j}{2}}\|\Delta_j^v(a_F \cdot \nabla a_F)\|_{L_T^q L^{\frac{p}{2}}(\mathbb{R}^3)} \lesssim d_j \nu_h^{-\frac{1}{q}} 2^{(3-\frac{4}{p}-\frac{2}{q})j}\|a\|^2_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}, \quad q < \frac{p}{p-2},$$

where $(d_k)_{k\in\mathbb{Z}}$ denotes a generic element of the sphere of $l^1(\mathbb{Z})$. From $3 - \frac{4}{p} - \frac{2}{q} = 0$ and $q \geq 1$, we have $p \leq 4$. Thus, we think $p = 4$ seems a special point.

The following proposition, which will be proved in Section 7, shows that Theorem 1.1 can be applied to initial data given by (1.3).

Proposition 1.2. *Let $\phi(x_h)$ and $\psi(x_h)$ be in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, supp$\hat{\phi}$, supp$\hat{\psi} \subset \mathcal{C}_h$, where $\mathcal{C}_h$ is a ring of $\mathbb{R}^2$. Denote $\phi_\varepsilon(x_h) = e^{ix_1/\varepsilon}\phi(x_h)$ and $\psi_\varepsilon(x_h) = e^{ix_1/\varepsilon}\psi(x_h)$, we have, for any positive ε, $q \in (1,\infty]$, $\sigma > 0$, $\alpha \in (0, 2(1-\frac{1}{q}))$,*

$$\|\phi_\varepsilon\|_{\tilde{B}_{q,1}^{-\sigma}} \leq C_\phi \varepsilon^\sigma,$$

$$\|\phi_\varepsilon\|_{\dot{B}_{q,1}^{-\alpha}} \leq C_\phi \varepsilon^\alpha, \tag{1.6}$$

$$\|\phi_\varepsilon\|_{\dot{B}_{q,\infty}^{-\sigma}} \geq C_\phi \varepsilon^\sigma,$$

$$\int_0^\infty \|\phi_{\varepsilon,F'}\psi_{\varepsilon,F'}\|_{L^2(\mathbb{R}^2)} dt \leq C_{\phi,\psi,\nu_h}\varepsilon^2,$$

where

$$\phi_{\varepsilon,F'}(t) = e^{\nu_h t \Delta_h}\sum_{k\geq 0}\Delta_k^h \phi_\varepsilon, \quad \psi_{\varepsilon,F'}(t) = e^{\nu_h t \Delta_h}\sum_{k\geq 0}\Delta_k^h \psi_\varepsilon,$$

$$\|a\|_{\tilde{B}_{q,1}^{-\sigma}} = \|S_0^h a\|_{L^2} + \sum_{k=0}^\infty 2^{-\sigma k}\|\Delta_k^h a\|_{L^q}, \quad \|a\|_{\dot{B}_{q,1}^{-\alpha}} = \sum_{k\in\mathbb{Z}} 2^{-\alpha k}\|\Delta_k^h a\|_{L^q}$$

and $\|a\|_{\dot{B}_{q,\infty}^{-\sigma}} = \sup_{k\in\mathbb{Z}} 2^{-\sigma k}\|\Delta_k^h a\|_{L^q}$.

Remark 1.4. From Proposition 1.2, we get

$$\|u_0^\varepsilon\|_{B_4^{-\frac{1}{2},\frac{1}{2}}} \simeq C_{\phi_0,\phi_1}, \quad \text{and} \quad [u_0^\varepsilon]_{E_\infty^p} \leq C_{\phi_0,\phi_1,\nu_h}\varepsilon^{\frac{1}{2}-\frac{2}{p}}, \quad \text{for all } p > 4.$$

Theorem 1.1 and Proposition 1.2 imply that the anisotropic Navier-Stokes system (1.1) with initial data $u_0^{\varepsilon,q}$, which defined by

$$u_0^{\varepsilon,q}(x) = \varepsilon^{-1+\frac{2}{q}}\sin(\frac{x_1}{\varepsilon})(0, -\partial_{x_3}(\phi_0\phi_1), \partial_{x_2}(\phi_0\phi_1)), \quad \forall\, q \geq 2,$$

is globally wellposed when ε is small enough.

At last, we give an imbedding result in the following proposition, which will be proved in Section 8.

Proposition 1.3. *For $p \geq 2$, we have*

$$B^{0,\frac{1}{2}} \subset B_p^{-1+\frac{2}{p},\frac{1}{2}} \subset \dot{B}_{\infty,2}^{-1} \subset BMO^{-1} \subset \dot{B}_{\infty,\infty}^{-1} = C^{-1},$$

where $\|f\|_{\dot{B}_{\infty,q}^{-1}} = \big\|2^{-k}\|\Delta_k f\|_{L^\infty}\big\|_{l_k^q}$, $\Delta_k a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a})$,

$$\|f\|_{BMO^{-1}} := \|f\|_{\dot{B}_{\infty,\infty}^{-1}} + \sup_{x\in\mathbb{R}^3, R>0} R^{-\frac{3}{2}}\left(\int_{P(x,R)} |e^{t\Delta}f(y)|^2 dy dt\right)^{\frac{1}{2}},$$

$P(x,R) = [0, R^2] \times B(x,R)$ and $B(x,R) := \{y \in \mathbb{R}^3; |x-y| \leq R\}$ (see [11]).

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1.3 Structure of the proof of Theorem 1.1.

The purpose of Section 2 is to establish some results about anisotropic Littlewood-Paley theory, which will be of constant use in what follows.

Section 3 will be devoted to the proof of the existence of a solution of (1.1). In order to do it, we shall search for a solution of the form, (following the idea in [3])

$$u = u_F + w, \quad \text{and} \quad w \in B^{0,\frac{1}{2}}(\infty).$$

In Section 4, we shall prove the uniqueness in the following way. First, we shall establish a regularity theorem, which claim that if $u \in B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ is a solution of (1.1) with $[u_0]_{E_T^p} < \infty$, then there exists $T_1 \in (0,T]$ such that $w = u - u_F \in B^{0,\frac{1}{2}}(T_1)$. Therefore, looking at the equation of w, we shall prove the uniqueness of the solution u in the space $u_F + B^{0,\frac{1}{2}}(T_1)$. Since $u \in C([0,T]; B_p^{-1+\frac{2}{p},\frac{1}{2}})$, one can easily obtain the uniqueness of the solution u on $[0,T]$.

In Section 5, we shall prove that if $\nu_3 > 0$, then the continuous dependence of the solution on the initial data holds.

We should mention that the methods introduced by Chemin-Gallagher in [2], Chemin-Zhang in [3], Koch-Tataru in [11] and Paicu in [12] will play a crucial role in our proof here.

2 Anisotropic Littlewood-Paley theory

At first, we list anisotropic Berstein inequalities in the following, (please see the detail in [3, 12]).

Lemma 2.1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) be a ball of $\mathbb{R}_h^2$ (resp. $\mathbb{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) be a ring of $\mathbb{R}_h^2$ (resp. $\mathbb{R}_v$). Then, for $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$, there holds:*

1. *If the support of $\hat{a}$ is included in $2^k \mathcal{B}_h$, then*

$$\|\partial_h^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\beta|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^{q_1})},$$

   where $\partial_h := \partial_{x_h}$.

2. *If the support of $\hat{a}$ is included in $2^l \mathcal{B}_v$, then*

$$\|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{l(N+\frac{1}{q_2}-\frac{1}{q_1})} \|a\|_{L_h^{p_1}(L_v^{q_2})},$$

   where $\partial_3 := \partial_{x_3}$.

3. *If the support of $\hat{a}$ is included in $2^k \mathcal{C}_h$, then*

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\beta|=N} \|\partial_h^\beta a\|_{L_h^{p_1}(L_v^{q_1})}.$$

4. *If the support of $\hat{a}$ is included in $2^l \mathcal{C}_v$, then*

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-lN} \|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

Let us state two corollaries of this lemma, the proofs of which are obvious and thus omitted.

Corollary 2.1. *The space $B^{0,\frac{1}{2}}$ is continuously embedded in the space $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ and so is $B^{0,\frac{1}{2}}(T)$ in $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$, $p \geq 2$. Moreover, the space $B^{0,\frac{1}{2}}(T)$ is continuously embedded in the space $L_T^\infty(L_h^2(L_v^\infty))$. Furthermore, The space $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ is continuously embedded in the space $B_{2p}^{-1+\frac{1}{p},\frac{1}{2}}$ and so is $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ in $B_{2p}^{-1+\frac{1}{p},\frac{1}{2}}(T)$, $p \geq 2$.*

Corollary 2.2. *If a belongs to $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$, $p \geq 2$, then we have*

$$\sum_{l\in\mathbb{Z}} 2^{\frac{l}{2}} \left(\sum_{k\in\mathbb{Z}} 2^{2k(-1+\frac{2}{p})} \|\Delta_k^h \Delta_l^v a(0)\|_{L_h^p(L_v^2)}^2 \right)^{\frac{1}{2}} \lesssim \|a(0)\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

and

$$\sum_{l\in\mathbb{Z}} 2^{\frac{l}{2}} \left[\sum_{k\in\mathbb{Z}} \left(2^{k(-2+\frac{4}{p})} \|\Delta_k^h \Delta_l^v a\|_{L_T^\infty(L_h^p(L_v^2))}^2 + \nu_h 2^{\frac{4k}{p}} \|\Delta_k^h \Delta_l^v a\|_{L_T^2(L_h^p(L_v^2))}^2 \right) \right]^{\frac{1}{2}} \lesssim \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)}.$$

Notations. In what follows, as in [3], we make the convention that $(c_k)_{k\in\mathbb{Z}}$ (resp. $(d_k)_{k\in\mathbb{Z}}$) denotes a generic element of the sphere of $l^2(\mathbb{Z})$ (resp. $l^1(\mathbb{Z})$). Moreover, $(c_{k,l})_{(k,l)\in\mathbb{Z}^2}$ denotes a generic element of the sphere of $l^2(\mathbb{Z}^2)$ and $(d_{k,l})_{(k,l)\in\mathbb{Z}^2}$ denotes a generic sequence such that

$$\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} d_{k,l}^2 \right)^{\frac{1}{2}} = 1.$$

The following lemma will be of frequent use in this work. It describes some estimates of dyadic parts of functions in $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$.

Lemma 2.2. *For any $a \in B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)$, $p' \geq 2$, we have*

$$S_{k,l}(a) \lesssim \frac{p'}{\beta} d_{k,l} 2^{\frac{\beta k}{p'}} 2^{-\frac{l}{2}} \|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)}$$

and

$$S_k(a) \lesssim \frac{p'}{\beta} c_k 2^{\frac{\beta k}{p'}} \|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)},$$

where $\beta > 0$, and

$$S_{k,l}(a) = \sum_{k'\leq k-1} 2^{(-1+\frac{2+\beta}{p'})k'} \left(\|\Delta_{k'}^h \Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))} + \nu_h^{\frac{1}{2}} \|\nabla_h \Delta_{k'}^h \Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))} \right),$$

$$S_k(a) := \sum_{k'\leq k-1} 2^{(-1+\frac{2+\beta}{p'})k'} \left(\|\Delta_{k'}^h a\|_{L_T^\infty(L_h^{p'}(L_v^\infty))} + \nu_h^{\frac{1}{2}} \|\nabla_h \Delta_{k'}^h a\|_{L_T^2(L_h^{p'}(L_v^\infty))} \right).$$

Proof. Since

$$2^{\frac{l}{2}} 2^{-\frac{\beta k}{p'}} S_{k,l} \;\leq\; 2^{\frac{l}{2}} \sum_{k'\leq k-1} 2^{-\frac{\beta}{p'}(k-k')} 2^{(-1+\frac{2}{p'})k'} \left(\|\Delta_{k'}^h \Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))} \right.$$

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$$+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\Bigg),$$

using Young's inequality, we obtain

$$2^{\frac{l}{2}}\left(\sum_{k\in\mathbb{Z}}2^{-\frac{2\beta k}{p'}}S_{k,l}^2\right)^{\frac{1}{2}}\lesssim 2^{\frac{l}{2}}\frac{p'}{\beta}\left(\sum_{k'\in\mathbb{Z}}2^{(-2+\frac{4}{p'})k'}\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right.\right.$$

$$\left.\left.+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^2\right)^{\frac{1}{2}}.$$

Combining it with Corollary 2.2, we can easily obtain the first inequality.

To get the second inequality, we shall prove that, for any $(c_k)_{k\in\mathbb{Z}}$, we have

$$I(a):=\sum_{k\in\mathbb{Z}}2^{-\frac{\beta k}{p'}}S_k(a)c_k\lesssim\frac{p'}{\beta}\|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)}.\tag{2.1}$$

Using Lemma 2.1, we get

$$S_k(a)\lesssim\sum_{k'\leq k-1}\sum_{l\in\mathbb{Z}}2^{\frac{l}{2}+(-1+\frac{2+\beta}{p'})k'}\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right),$$

and

$$I(a)\lesssim\sum_{l\in\mathbb{Z}}2^{\frac{l}{2}}\sum_{\substack{(k,k')\in\mathbb{Z}^2\\k'\leq k-1}}2^{-\frac{\beta}{p'}(k-k')}2^{(-1+\frac{2}{p'})k'}c_k\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right.$$

$$\left.+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right).$$

Using the Cauchy-Schwarz inequality and Young's inequality, we have

$$I(a)$$
$$\lesssim\left(\sum_{\substack{(k,k')\in\mathbb{Z}^2\\k'\leq k-1}}2^{-\frac{\beta}{p'}(k-k')}c_k^2\right)^{\frac{1}{2}}\sum_{l\in\mathbb{Z}}2^{\frac{l}{2}}\left(\sum_{\substack{(k,k')\in\mathbb{Z}^2\\k'\leq k-1}}2^{-\frac{\beta}{p'}(k-k')}2^{(-2+\frac{4}{p'})k'}\right.$$

$$\left.\times\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^2\right)^{\frac{1}{2}}$$

$$\lesssim\sqrt{\frac{p'}{\beta}}\sum_{l\in\mathbb{Z}}2^{\frac{l}{2}}\left(\sum_{\substack{(k,k')\in\mathbb{Z}^2\\k'\leq k-1}}2^{-\frac{\beta}{p'}(k-k')}2^{(-2+\frac{4}{p'})k'}\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right.\right.$$

$$\left.\left.+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^2\right)^{\frac{1}{2}}$$

$$\lesssim\frac{p'}{\beta}\sum_{l\in\mathbb{Z}}2^{\frac{l}{2}}\left(\sum_{k'\in\mathbb{Z}}2^{(-2+\frac{4}{p'})k'}\left(\|\Delta_{k'}^h\Delta_l^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}+\nu_h^{\frac{1}{2}}\|\nabla_h\Delta_{k'}^h\Delta_l^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^2\right)^{\frac{1}{2}}$$

$$\lesssim\frac{p'}{\beta}\|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)},$$

which proves (2.1) and thus Lemma 2.2. $\qquad\qquad\square$

With Lemma 2.2, we are going to state two lemmas, which is very closed to Sobolev's embedding Theorem and will be of constant use in the proof of Theorem 1.1.

Lemma 2.3. *The space* $B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)$ *is included in* $L_T^{\frac{2p'}{p'-2}}(L_h^{p'}(L_v^\infty))$, $p' > 2$. *Moreover precisely, if* $a \in B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)$, *then, we have*

$$\|\Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2}}(L_h^{p'}(L_v^2))} \lesssim \sqrt{\frac{p'}{\beta}} d_j \nu_h^{\frac{1}{p'}-\frac{1}{2}} 2^{-\frac{j}{2}} \|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)} \tag{2.2}$$

and

$$\|a\|_{L_T^{\frac{2p'}{p'-2}}(L_h^{p'}(L_v^\infty))} \lesssim \|a\|_{\widetilde{L}_T^{\frac{2p'}{p'-2}}(\widetilde{L}_h^{p'}(B_v^{\frac{1}{2}}))} \lesssim \sqrt{\frac{p'}{\beta}} \nu_h^{\frac{1}{p'}-\frac{1}{2}} \|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac{1}{2}}(T)}, \tag{2.3}$$

where $\|a\|_{\widetilde{L}_T^{\frac{2p'}{p'-2}}(\widetilde{L}_h^{p'}(B_v^{\frac{1}{2}}))} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2}}(L_h^{p'}(L_v^2))}$ *and* $\beta \in (0, \min\{p'-2, 2\}]$. *Furthermore, for* $p \geq 2$, *we have*

$$\|\Delta_j^v a\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))} \lesssim d_j \nu_h^{\frac{1}{2p}-\frac{1}{2}} 2^{-\frac{j}{2}} \|a\|_{B_{2p}^{-1+\frac{1}{p},\frac{1}{2}}(T)} \lesssim d_j \nu_h^{\frac{1}{2p}-\frac{1}{2}} 2^{-\frac{j}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)} \tag{2.4}$$

and

$$\|a\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \lesssim \|a\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \lesssim \nu_h^{\frac{1}{2p}-\frac{1}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)}, \tag{2.5}$$

where $\|a\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} = \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v a\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))}$.

Remark 2.1. From now on, $A \lesssim B$ means $A \leq C(p)B$, where $C(p)$ is a constant depending on p.

Proof. Let us first notice that

$$\|\Delta_j^v a\|^2_{L_T^{\frac{2p'}{p'-2}}(L_h^{p'}(L_v^2))} = \|(\Delta_j^v a)^2\|_{L_T^{\frac{p'}{p'-2}}(L_h^{\frac{p'}{2}}(L_v^1))}.$$

Using Bony's decomposition in the horizontal variable, we have

$$(\Delta_j^v a)^2 = \sum_{k\in\mathbb{Z}} S_{k-1}^h \Delta_j^v a \Delta_k^h \Delta_j^v a + \sum_{k\in\mathbb{Z}} S_{k+2}^h \Delta_j^v a \Delta_k^h \Delta_j^v a.$$

These two terms are estimated exactly in the same way. Applying Hölder's inequality and Lemma 2.1, we obtain

$$\|S_{k-1}^h \Delta_j^v a \Delta_k^h \Delta_j^v a\|_{L_T^{\frac{p'}{p'-2}}(L_h^{\frac{p'}{2}}(L_v^1))}$$

$$\lesssim \|S_{k-1}^h \Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2-\beta}}(L_h^{p'}(L_v^2))} \|\Delta_k^h \Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2+\beta}}(L_h^{p'}(L_v^2))}$$

$$\lesssim \sum_{k'\leq k-2} \|\Delta_{k'}^h \Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2-\beta}}(L_h^{p'}(L_v^2))} \|\Delta_k^h \Delta_j^v a\|_{L_T^{\frac{2p'}{p'-2+\beta}}(L_h^{p'}(L_v^2))}$$

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$$\lesssim \sum_{k'\leq k-2}\|\Delta_{k'}^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}^{1-\frac{2+\beta}{p'}}\|\Delta_{k'}^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}^{\frac{2+\beta}{p'}}\|\Delta_k^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}^{1-\frac{2-\beta}{p'}}\|\Delta_k^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}^{\frac{2-\beta}{p'}}$$

$$\lesssim \sum_{k'\leq k-2}\left(2^{\frac{2+\beta}{p'}k'}\|\Delta_{k'}^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^{1-\frac{2+\beta}{p'}}\left(2^{(-1+\frac{2+\beta}{p'})k'}\|\Delta_{k'}^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right)^{\frac{2+\beta}{p'}}$$

$$\times\left(2^{\frac{2-\beta}{p'}k}\|\Delta_k^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}\right)^{1-\frac{2-\beta}{p'}}\left(2^{(-1+\frac{2-\beta}{p'})k}\|\Delta_k^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right)^{\frac{2-\beta}{p'}}$$

$$\lesssim \nu_h^{\frac{2}{p'}-1}\sum_{k'\leq k-2}2^{(-1+\frac{2+\beta}{p'})k'}\left(\nu_h^{\frac12}\|\nabla_h\Delta_{k'}^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}+\|\Delta_{k'}^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right)$$

$$\times 2^{(-1+\frac{2-\beta}{p'})k}\left(\nu_h^{\frac12}\|\nabla_h\Delta_k^h\Delta_j^v a\|_{L_T^2(L_h^{p'}(L_v^2))}+\|\Delta_k^h\Delta_j^v a\|_{L_T^\infty(L_h^{p'}(L_v^2))}\right).$$

From Corollary 2.2 and Lemma 2.2, we get

$$\|S_{k-1}^h\Delta_j^v a\Delta_k^h\Delta_j^v a\|_{L_T^{\frac{p'}{p'-2}}(L_h^{\frac{p'}{2}}(L_v^1))}\lesssim \frac{p'}{\beta}d_{k,j}^2\nu_h^{\frac{2}{p'}-1}2^{-j}\|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac12}(T)}^2.$$

Taking the sum over k, we obtain

$$\|(\Delta_j^v a)^2\|_{L_T^{\frac{p'}{p'-2}}(L_h^{\frac{p'}{2}}(L_v^1))}\lesssim \frac{p'}{\beta}d_j^2\nu_h^{\frac{2}{p'}-1}2^{-j}\|a\|_{B_{p'}^{-1+\frac{2}{p'},\frac12}(T)}^2,$$

which is exactly the first inequality of this lemma. Combining it with Lemma 2.1 and Corollary 2.1, we can immediately obtain (2.3)-(2.5). $\qquad\square$

Lemma 2.4. *Let a be in $B^{0,\frac12}(T)$. Then, we have*

$$\|\Delta_j^v a\|_{L_T^{q_1}(L_h^{q_2}(L_v^2))}\lesssim d_j\nu_h^{-\frac{1}{q_1}}2^{-\frac{j}{2}}\|a\|_{B^{0,\frac12}(T)},\quad \frac{1}{q_1}+\frac{1}{q_2}=\frac12,\ q_2\in[2,4],\qquad (2.6)$$

$$\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}\lesssim d_j\nu_h^{-\frac{1}{2p}}2^{-\frac{j}{2}}\|a\|_{B^{0,\frac12}(T)}$$

and

$$\|a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^\infty))}\lesssim \|a\|_{\widetilde L_T^{2p}(\widetilde L_h^{\frac{2p}{p-1}}(B_v^{\frac12}))}\lesssim \nu_h^{-\frac{1}{2p}}\|a\|_{B^{0,\frac12}(T)},$$

where $\|a\|_{\widetilde L_T^{2p}(\widetilde L_h^{\frac{2p}{p-1}}(B_v^{\frac12}))}=\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$ *and $p\geq 2$.*

Proof. From Corollary 2.1, we have

$$\|a\|_{B_4^{-\frac12,\frac12}(T)}\lesssim \|a\|_{B^{0,\frac12}(T)}.$$

From (2.2), we get

$$\|\Delta_j^v a\|_{L_T^4(L_h^4(L_v^2))}\lesssim d_j\nu_h^{-\frac14}2^{-\frac{j}{2}}\|a\|_{B_4^{-\frac12,\frac12}(T)}\lesssim d_j\nu_h^{-\frac14}2^{-\frac{j}{2}}\|a\|_{B^{0,\frac12}(T)}.$$

Combining it with

$$\|\Delta_j^v a\|_{L_T^\infty(L_h^2(L_v^2))}\lesssim d_j2^{-\frac{j}{2}}\|a\|_{B^{0,\frac12}(T)},$$

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using interpolation, we obtain (2.6). Choosing $q_1 = 2p$, we can finish the proof of this lemma.
□

Using Lemma 2.1, we can obtain some estimates of u_F in the following lemma.

Lemma 2.5. *Let* $u_0 \in B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \geq 2$, *and* u_F *be as in (1.4),* $1 \leq q \leq \infty$. *Then, there holds*

$$\|\Delta_k^h \Delta_l^v u_F\|_{L_T^q(L_h^p(L_v^2))}$$
$$\lesssim d_{k,l} 2^{(1-\frac{2}{p})k} 2^{-\frac{l}{2}} \min(\nu_h^{-\frac{1}{q}} 2^{-\frac{2k}{q}}, \nu_3^{-\frac{1}{q}} 2^{-\frac{2l}{q}}) \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}, \ for \ k \geq l-1, \quad (2.7)$$

$$\|\Delta_k^h \Delta_l^v u_F\|_{L_T^q(L_h^p(L_v^2))} = 0, \ \ for \ k < l-1. \quad (2.8)$$

Moreover, u_F *belongs to* $B_p^{-1+\frac{2}{p},\frac{1}{2}}(\infty) \cap C([0,\infty); B_p^{-1+\frac{2}{p},\frac{1}{2}})$, *and we have*

$$\|u_F\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(\infty)} \lesssim \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}. \quad (2.9)$$

Proof. The relation (2.5) in [3] tell us

$$\Delta_k^h \Delta_l^v u_F(t,x) = 2^{2k+l} \int_{\mathbb{R}^3} g_h(t, 2^k(x_h - y_h)) g_v(t, 2^l(x_3 - y_3)) \Delta_k^h \Delta_l^v u_0(y) dy. \quad (2.10)$$

with

$$\|g_h(t,\cdot)\|_{L^1(\mathbb{R}^2)} \lesssim e^{-c\nu_h t 2^{2k}}, \ \ \|g_v(t,\cdot)\|_{L^1(\mathbb{R})} \lesssim e^{-c\nu_3 t 2^{2l}}.$$

From Corollary 2.2 and (2.10), we have

$$\|\Delta_k^h \Delta_l^v u_F(t)\|_{L_h^p(L_v^2)} \lesssim e^{-c\nu_h t 2^{2k} - c\nu_3 t 2^{2l}} \|\Delta_k^h \Delta_l^v u_0\|_{L_h^p(L_v^2)}$$
$$\lesssim e^{-c\nu_h t 2^{2k} - c\nu_3 t 2^{2l}} d_{k,l} 2^{(1-\frac{2}{p})k} 2^{-\frac{l}{2}} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

By integration, we can obtain (2.7)-(2.9). The proof of $u_F \in C([0,\infty); B_p^{-1+\frac{2}{p},\frac{1}{2}})$ is simple, and we omit the details. □

From Lemmas 2.1 and 2.5, we can immediately deduce the following corollary.

Corollary 2.3. *For any* $(q, p') \in [1, \infty] \times [p, \infty]$, $p \geq 2$, *we have*

$$\|\Delta_k^h u_F\|_{L^q(\mathbb{R}^+; L_h^{p'}(L_v^\infty))} \lesssim \nu_h^{-\frac{1}{q}} c_k 2^{-k(\frac{2}{q} + \frac{2}{p'} - 1)} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

If $p' \in [p, \infty]$ *and* $q \in [1, \frac{2p'}{p'-2})$, *we have*

$$\|\Delta_j^v u_F\|_{L^q(\mathbb{R}^+; L_h^{p'}(L_v^2))} \lesssim \nu_h^{-\frac{1}{q}} d_j 2^{-j(\frac{2}{q} + \frac{2}{p'} - \frac{1}{2})} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

The following lemma is the end point of the second estimate of Corollary 2.3.

Lemma 2.6. *Under the assumptions of Lemma 2.5, we have*

$$\|\Delta_j^v u_F\|_{L^2(\mathbb{R}^+; L_h^\infty(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

and

$$\|u_F\|_{L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))} \lesssim \nu_h^{-\frac{1}{2}} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

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Proof. Trivially, there holds

$$\|\Delta_j^v u_F\|^2_{L^2_T(L_h^\infty(L_v^2))} = \|(\Delta_j^v u_F)^2\|_{L^1_T(L_h^\infty(L_v^1))}.$$

Using Bony's paradifferential decomposition in the horizontal variables, we have

$$(\Delta_j^v u_F)^2 = \sum_{k\in\mathbb{Z}} S_{k-1}^h \Delta_j^v u_F \Delta_k^h \Delta_j^v u_F + \sum_{k\in\mathbb{Z}} S_{k+2}^h \Delta_j^v u_F \Delta_k^h \Delta_j^v u_F. \qquad (2.11)$$

Using Lemma 2.1 and Hölder's inequality, we get

$$\|S_{k-1}^h \Delta_j^v u_F \Delta_k^h \Delta_j^v u_F\|_{L^1_T(L_h^\infty(L_v^1))} \lesssim 2^{\frac{2}{p}k} \|S_{k-1}^h \Delta_j^v u_F\|_{L^\infty_T(L_h^{2p}(L_v^2))} \|\Delta_k^h \Delta_j^v u_F\|_{L^1_T(L_h^{2p}(L_v^2))}$$

By (2.7) and the proof of Lemma 2.2, we obtain

$$\|S_{k-1}^h \Delta_j^v u_F\|_{L^\infty_T(L_h^{2p}(L_v^2))} \lesssim d_{k,j} 2^{(1-\frac{1}{p})k} 2^{-\frac{j}{2}} \|u_0\|_{B_{2p}^{-1+\frac{2}{2p},\frac{1}{2}}}.$$

Therefore, using (2.7) once again, we get

$$\|\sum_{k\in\mathbb{Z}} S_{k-1}^h \Delta_j^v u_F \Delta_k^h \Delta_j^v u_F\|_{L^1_T(L_h^\infty(L_v^1))} \lesssim 2^{-j} \nu_h^{-1} \left(\sum_{k\in\mathbb{Z}} d_{k,j}^2\right) \|u_0\|^2_{B_{2p}^{-1+\frac{2}{2p},\frac{1}{2}}}.$$

A similar argument yields a similar estimate for the other term in (2.11). Then we deduce that

$$\|\Delta_j^v u_F\|_{L^2(\mathbb{R}^+;L_h^\infty(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|u_0\|_{B_{2p}^{-1+\frac{2}{2p},\frac{1}{2}}} \lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

From Lemma 2.1, we conclude that

$$\|u_F\|_{L^2_T(L^\infty(\mathbb{R}^3))} \lesssim \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\Delta_j^v u_F\|_{L^2_T(L_h^\infty(L_v^2))} \lesssim \nu_h^{-\frac{1}{2}} \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}.$$

$\square$

3   The proof of an existence theorem

The purpose of this section is to prove the following existence theorem.

Theorem 3.1. *A sufficiently small constant c exists which satisfies the following property: if the divergence free vector field $u_0 \in B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $p \geq 2$ and $[u_0]_{E_\infty^p} \leq c\nu_h$, then the system (1.1) with initial data u_0 has a global solution in the space $\{u_F + B^{0,\frac{1}{2}}(\infty)\} \cap C([0,\infty); B_p^{-1+\frac{2}{p},\frac{1}{2}})$.*

Proof. As announced in the introduction, we shall look for a solution of the form

$$u = u_F + w.$$

Actually, by substituting the above formula to (1.1), we get

$$\begin{cases} w_t + w \cdot \nabla w - \nu_h \Delta_h w - \nu_3 \partial_3^2 w + w \cdot \nabla u_F + u_F \cdot \nabla w = -u_F \cdot \nabla u_F - \nabla P, \\ \operatorname{div} w = 0, \\ w|_{t=0} = u_{0ll} = u_0 - u_{0hh}, \end{cases} \qquad (3.1)$$

where

$$u_{0ll} = \sum_{j \in \mathbb{Z}} S_{j-1}^h \Delta_j^v u_0. \tag{3.2}$$

Moreover, we obtain $\|u_{0ll}\|_{B^{0,\frac{1}{2}}} \lesssim \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$. We shall use the classical Friedrichs' regularization method to construct the approximate solutions to (3.1). For simplicity, we just outline it here (for the details, see [3, 4, 12]). In order to do so, let us define the sequence of operators $(P_n)_{n \in \mathbb{N}}$, $(P_{1n})_{n \in \mathbb{N}}$ and $(P_{2n})_{n \in \mathbb{N}}$ by

$$P_n a := \mathcal{F}^{-1}(1_{B(0,n)}\hat{a}), \ P_{1n} a := \mathcal{F}^{-1}(1_{\{|\xi| \leq n, \ |\xi_3| \geq \frac{1}{n}\}}\hat{a}), \ P_{2n} a := \mathcal{F}^{-1}(1_{\{|\xi_3| < \frac{1}{n}\}}\hat{a}),$$

and we define the following approximate system:

$$\begin{cases} \partial_t w_n + P_n(w_n \cdot \nabla w_n) - \nu_h \Delta_h w_n - \nu_3 \partial_3^2 w_n + P_n(w_n \cdot \nabla u_F) + P_n(u_F \cdot \nabla w_n) \\ = -P_{1n}(u_F \cdot \nabla u_F) - P_n \nabla(-\Delta)^{-1}\partial_j \partial_k \left((u_F^j + w_n^j)(u_F^k + w_n^k) - P_{2n}(u_F^j u_F^k) \right), \\ \mathrm{div} w_n = 0, \\ w_n|_{t=0} = P_n(u_{0ll}), \end{cases} \tag{3.3}$$

where $(-\Delta)^{-1}\partial_j \partial_k$ is defined precisely by

$$(-\Delta)^{-1}\partial_j \partial_k a := \mathcal{F}^{-1}(|\xi|^{-2}\xi_j \xi_k \hat{a}).$$

Then, the system (3.1) appears to be an ordinary differential equation in the space

$$L_n^2 := \{a \in L^2(\mathbb{R}^3) | \mathrm{div} a = 0, \ \mathrm{Supp}\hat{a} \subset B(0,n)\}.$$

Such system is globally wellposed because

$$\frac{d}{dt}\|w_n\|_{L^2}^2 \leq C_n\|u_F\|_{L^\infty}\|w_n\|_{L^2}^2 + C_n\|u_F \cdot \nabla u_F\|_{B^{0,\frac{1}{2}}}\|w_n\|_{L^2},$$

and u_F belongs to $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$.

Now, the proof of Theorem 3.1 reduces to the following three propositions, which we shall admit for the time begin.

Proposition 3.1. *Let a be a divergence free vector filed in $B^{0,\frac{1}{2}}(T)$ and u be a divergence free vector field in $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$F_j(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v(u \cdot \nabla a)\Delta_j^v a dx \right| dt \lesssim d_j^2 \nu_h^{-\frac{1}{2}-\frac{1}{2p}} 2^{-j}\|a\|_{B^{0,\frac{1}{2}}(T)}^2 \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Proposition 3.2. *Let a and b be two divergence free vector fields in $B^{0,\frac{1}{2}}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$G_j(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v(a \cdot \nabla u_F)\Delta_j^v b dx \right| dt$$

$$\lesssim d_j^2 2^{-j}\|a\|_{B^{0,\frac{1}{2}}(T)}\|b\|_{B^{0,\frac{1}{2}}(T)} \left(\nu_h^{-\frac{1}{2}-\frac{1}{2p}}\|u_F\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_T^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} \right),$$

where $\|u_F\|_{\widetilde{L}_T^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} := \sum_{j \in \mathbb{Z}} 2^{\frac{3}{2}j}\|\Delta_j^v u_F\|_{L_T^1(L_h^\infty(L_v^2))}$.

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From Corollary 2.1, Lemma 2.3 and Proposition 3.1, we have the following proposition.

Proposition 3.3. *Let a and b be two divergence free vector fields in $B^{0,\frac{1}{2}}(T)$. Then, for any $j \in \mathbb{Z}$, we have*

$$\int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v(a \cdot \nabla b) \Delta_j^v b \, dx \right| dt \lesssim d_j^2 \nu_h^{-1} 2^{-j} \|a\|_{B^{0,\frac{1}{2}}(T)} \|b\|_{B^{0,\frac{1}{2}}(T)}^2.$$

Conclusion of the proof of Theorem 3.1. Applying the operator Δ_j^v to (3.3) and taking the L^2 inner product of the resulting equation with $\Delta_j^v w_n$, we have

$$\frac{d}{dt}\|\Delta_j^v w_n\|_{L^2}^2 + 2\nu_h\|\nabla_h \Delta_j^v w_n\|_{L^2}^2 + 2\nu_3\|\partial_3 \Delta_j^v w_n\|_{L^2}^2$$

$$= -2\int_{\mathbb{R}^3} \Delta_j^v(w_n \cdot \nabla w_n)\Delta_j^v w_n dx - 2\int_{\mathbb{R}^3} \Delta_j^v(u_F \cdot \nabla w_n)\Delta_j^v w_n dx$$

$$-2\int_{\mathbb{R}^3} \Delta_j^v(w_n \cdot \nabla u_F)\Delta_j^v w_n dx - 2\int_{\mathbb{R}^3} \Delta_j^v P_{1n}(u_F \cdot \nabla u_F)\Delta_j^v w_n dx.$$

From Lemmas 2.3, 2.5, Corollary 2.3 and Propositions 3.1-3.3, we get

$$2^j \left(\|\Delta_j^v w_n\|_{L_T^\infty(L^2)}^2 + 2\nu_h\|\nabla_h \Delta_j^v w_n\|_{L_T^2(L^2)}^2 + 2\nu_3\|\partial_3 \Delta_j^v w_n\|_{L_T^2(L^2)}^2 \right)$$

$$\leq 2^j \|\Delta_j^v w_n(0)\|_{L^2}^2 + Cd_j^2 \nu_h^{-1} \|w_n\|_{B^{0,\frac{1}{2}}(T)}^2 \left(\|w_n\|_{B^{0,\frac{1}{2}}(T)} + \|u_0\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} \right)$$

$$+ C2^j \|\Delta_j^v(u_F \cdot \nabla u_F)\|_{L_T^1(L^2)} \|\Delta_j^v w_n\|_{L_T^\infty(L^2)}$$

and

$$\|w_n\|_{B^{0,\frac{1}{2}}(T)}$$

$$\leq 2C_0[u_0]_{E_\infty^p} + C\nu_h^{-\frac{1}{2}} \|w_n\|_{B^{0,\frac{1}{2}}(T)} \left(\|w_n\|_{B^{0,\frac{1}{2}}(T)} + [u_0]_{E_\infty^p} \right)^{\frac{1}{2}}.$$

Then, we have

$$\|w_n\|_{B^{0,\frac{1}{2}}(T)} \leq 2C_0[u_0]_{E_\infty^p} + \frac{4CC_0\sqrt{4C_0+1}}{\sqrt{\nu_h}}[u_0]_{E_\infty^p}^{\frac{3}{2}},$$

for all $T < T_n := \sup\{t > 0; \|w_n\|_{B^{0,\frac{1}{2}}(t)} \leq 4C_0[u_0]_{E_\infty^p}\}$. Then, if $[u_0]_{E_\infty^p}$ is small enough with respect to ν_h, we get for any n and for any $T < T_n$,

$$\|w_n\|_{B^{0,\frac{1}{2}}(T)} \leq \frac{5}{2}C_0[u_0]_{E_\infty^p}.$$

Thus, $T_n = +\infty$. Then, the existence follows from classical compactness method, the details of which are omitted (see [4, 12]).

In order to prove the continuity of the solution u, we have to prove the continuity of w. From (3.1), we have

$$\Delta_j^v w_t = \nu_h \Delta_j^v \Delta_h w + \nu_3 \Delta_j^v \partial_3^2 w - \Delta_j^v(w \cdot \nabla w)$$
$$-\Delta_j^v(w \cdot \nabla u_F) - \Delta_j^v(u_F \cdot \nabla w) - \Delta_j^v(u_F \cdot \nabla u_F) - \Delta_j^v \nabla P.$$

We can easily obtain that for any $T > 0$ and $j \in \mathbb{Z}$,

$$\nu_3 \Delta_j^v \partial_3^2 w \in L^\infty([0,T]; L^2), \quad \nu_h \Delta_j^v \Delta_h w \in L^2(0,T; L_v^2(\dot{H}_h^{-1})),$$

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$$\Delta_j^v(u_F \cdot \nabla u_F) \in L^1([0,T]; L^2)$$

and

$$(\nu_h \Delta_j^v \Delta_h w + \nu_3 \Delta_j^v \partial_3^2 w - \Delta_j^v(u_F \cdot \nabla u_F)|\Delta_j^v w)_{L^2} \in L^1([0,T]).$$

From Propositions 3.1-3.3, we have

$$(\Delta_j^v(w \cdot \nabla w) + \Delta_j^v(w \cdot \nabla u_F) + \Delta_j^v(u_F \cdot \nabla w)|\Delta_j^v w)_{L^2} \in L^1([0,T]).$$

Thus, we have $\frac{d}{dt}\|\Delta_j^v w(t)\|_{L^2}^2 \in L^1([0,T])$, for any $T > 0$ and $j \in \mathbb{Z}$. Combining it with $w \in B^{0,\frac{1}{2}}(\infty)$, we can easily get $w \in C([0,\infty); B^{0,\frac{1}{2}})$. Then Theorem 3.1 is proved provided of course that we have proved Propositions 3.1-3.2. $\square$

To prove Propositions 3.1-3.2, we need the following two lemmas.

Lemma 3.1. *Let a be in $B^{0,\frac{1}{2}}(T)$ and u be in $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$. We have*

$$\|\Delta_j^v(u\partial_h a)\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|a\|_{B^{0,\frac{1}{2}}(T)} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta_j^v(u\partial_h a) = \sum_{|j-j'|\leq 5} \Delta_j^v(S_{j'-1}^v u \partial_h \Delta_{j'}^v a) + \sum_{j'\geq j-N_0} \Delta_j^v(\Delta_{j'}^v u \partial_h S_{j'+2}^v a).$$

Using Hölder's inequality and Lemma 2.3, we get

$$\|\Delta_j^v(S_{j'-1}^v u \partial_h \Delta_{j'}^v a)\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))} \lesssim \|S_{j'-1}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \|\Delta_{j'}^v \partial_h a\|_{L_T^2(L^2(\mathbb{R}^3))}$$

$$\lesssim d_{j'} \nu_h^{-\frac{1}{2}} 2^{-\frac{j'}{2}} \|a\|_{B^{0,\frac{1}{2}}(T)} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}$$

and

$$\|\Delta_j^v(\Delta_{j'}^v u \partial_h S_{j'+2}^v a)\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))} \lesssim \|S_{j'+2}^v(\partial_h a)\|_{L_T^2(L_h^2(L_v^\infty))} \|\Delta_{j'}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))}$$

$$\lesssim d_{j'} \nu_h^{-\frac{1}{2}} 2^{-\frac{j'}{2}} \|a\|_{B^{0,\frac{1}{2}}(T)} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Then, we can immediately finish the proof. $\square$

Lemma 3.2. *Let a be in $B^{0,\frac{1}{2}}(T)$ and u be in $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$. We have*

$$\|\Delta_j^v(ua)\|_{L_T^2(L^2(\mathbb{R}^3))} \lesssim d_j \nu_h^{-\frac{1}{2p}} 2^{-\frac{j}{2}} \|a\|_{B^{0,\frac{1}{2}}(T)} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Proof. Using Bony's decomposition in the vertical variable, we obtain

$$\Delta_j^v(ua) = \sum_{|j-j'|\leq 5} \Delta_j^v(S_{j'-1}^v u \Delta_{j'}^v a) + \sum_{j'\geq j-N_0} \Delta_j^v(S_{j'+2}^v a \Delta_{j'}^v u).$$

Using Hölder's inequality, Lemmas 2.3-2.4, we get

$$\|\Delta_j^v(S_{j'-1}^v u \Delta_{j'}^v a)\|_{L_T^2(L^2(\mathbb{R}^3))} \lesssim \|S_{j'-1}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \|\Delta_{j'}^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$

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$$\lesssim \quad d_{j'}\nu_h^{-\frac{1}{2p}}2^{-\frac{j'}{2}}\|a\|_{B^{0,\frac{1}{2}}(T)}\|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}$$

and

$$\|\Delta_j^v(S_{j'+2}^v a\Delta_{j'}^v u)\|_{L_T^2(L^2(\mathbb{R}^3))} \quad \lesssim \quad \|S_{j'+2}^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^\infty))}\|\Delta_{j'}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))}$$

$$\lesssim \quad d_{j'}\nu_h^{-\frac{1}{2p}}2^{-\frac{j'}{2}}\|a\|_{B^{0,\frac{1}{2}}(T)}\|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Then, we can immediately finish the proof. $\qquad\square$

Proof of Proposition 3.1. We distinguish the terms with horizontal derivatives from the terms with vertical ones, writing

$$F_j(T) \leq F_j^h(T) + F_j^v(T),$$

where

$$F_j^h(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v(u^h \cdot \nabla_h a)\Delta_j^v a\, dx \right| dt$$

and

$$F_j^v(T) := \int_0^T \left| \int_{\mathbb{R}^3} \Delta_j^v(u^3\partial_3 a)\Delta_j^v a\, dx \right| dt.$$

Using Hölder's inequality, Lemmas 2.4 and 3.1, we obtain

$$F_j^h(T) \quad \leq \quad \|\Delta_j^v(u^h \cdot \nabla_h a)\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))}\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$

$$\lesssim \quad d_j^2\nu_h^{-\frac{1}{2}-\frac{1}{2p}}2^{-j}\|a\|_{B^{0,\frac{1}{2}}(T)}^2\|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

Applying the trick from [3, 12], using paradifferential decomposition in the vertical variable to $\Delta_j^v(u^3\partial_3 a)$ first, then by a commutator process, one get

$$\Delta_j^v(u^3\partial_3 a) \quad = \quad S_{j-1}^v u^3\partial_3\Delta_j^v a + \sum_{|j-l|\leq 5}[\Delta_j^v; S_{l-1}^v u^3]\partial_3\Delta_l^v a$$

$$+ \sum_{|j-l|\leq 5}(S_{l-1}^v u^3 - S_{j-1}^v u^3)\partial_3\Delta_j^v\Delta_l^v a + \sum_{l\geq j-N_0}\Delta_j^v(\Delta_l^v u^3\partial_3 S_{l+2}^v a).$$

Correspondingly, we decompose $F_j^v(T)$ as

$$F_j^v(T) := F_j^{1,v}(T) + F_j^{2,v}(T) + F_j^{3,v}(T) + F_j^{4,v}(T).$$

Using integration by parts and the fact that div$u$=0, we have

$$F_j^{1,v}(T) = \frac{1}{2}\int_0^T \left| \int_{\mathbb{R}^3} S_{j-1}^v \text{div}_h u^h|\Delta_j^v a|^2 dx \right| dt = \int_0^T \left| \int_{\mathbb{R}^3} S_{j-1}^v u^h \cdot \nabla_h\Delta_j^v a\Delta_j^v a\, dx \right| dt.$$

From Lemmas 2.3-2.4 and Hölder's inequality, we get

$$F_j^{1,v}(T) \quad \leq \quad \|S_{j-1}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))}\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}\|\nabla_h\Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))}$$

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$$\lesssim d_j^2 \nu_h^{-\frac{1}{2}-\frac{1}{2p}} 2^{-j} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \|a\|^2_{B^{0,\frac{1}{2}}(T)}.$$

To deal with the commutator in $F_j^{2,v}$, we first use the Taylor formula to get

$$F_j^{2,v}(T) = \sum_{|j-l|\leq 5} \int_0^T \left| \int_{\mathbb{R}^3} 2^j \int_{\mathbb{R}} h(2^j(x_3-y_3)) \int_0^1 S_{l-1}^v \partial_3 u^3(x_h, \tau y_3 + (1-\tau)x_3)d\tau \right.$$
$$\left. \times (y_3-x_3)\partial_3 \Delta_l^v a(x_h, y_3)dy_3 \Delta_j^v a(x)dx \right| dt.$$

Using div$u = 0$ and integration by parts, we have

$$F_j^{2,v}(T) = \sum_{|j-l|\leq 5} \int_0^T \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}} \bar{h}(2^j(x_3-y_3)) \int_0^1 S_{l-1}^v u^h(x_h, \tau y_3 + (1-\tau)x_3)d\tau \right.$$
$$\left. \cdot \nabla_h \partial_3 \Delta_l^v a(x_h, y_3)dy_3 \Delta_j^v a(x)dx \right| dt$$
$$+ \sum_{|j-l|\leq 5} \int_0^T \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}} \bar{h}(2^j(x_3-y_3)) \int_0^1 S_{l-1}^v u^h(x_h, \tau y_3 + (1-\tau)x_3)d\tau \right.$$
$$\left. \times \partial_3 \Delta_l^v a(x_h, y_3)dy_3 \nabla_h \Delta_j^v a(x)dx \right| dt,$$

where $\bar{h}(x_3) = x_3 h(x_3)$. Using Hölder's inequality, Young's inequality and Lemma 2.6, we obtain

$$F_j^{2,v}(T) \lesssim \sum_{|j-l|\leq 5} 2^{l-j}\|S_{l-1}^v u^h\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \|\nabla_h \Delta_l^v a\|_{L_T^2(L^2(\mathbb{R}^3))} \|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$
$$+ \sum_{|j-l|\leq 5} 2^{l-j}\|S_{l-1}^v u^h\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \|\nabla_h \Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))} \|\Delta_l^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$
$$\lesssim d_j^2 \nu_h^{-\frac{1}{2}-\frac{1}{2p}} 2^{-j} \|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \|a\|^2_{B^{0,\frac{1}{2}}(T)}.$$

It is easy to see that

$$F_j^{3,v}(T) \leq \sum_{\substack{|j-l'|\leq 5 \\ |j-l|\leq 5}} \int_0^T \left| \int_{\mathbb{R}^3} \Delta_{l'}^v u^3 \partial_3 \Delta_j^v \Delta_l^v a \Delta_j^v a dx \right| dt.$$

We can rewrite $\Delta_{l'}^v u^3$ as following:

$$\Delta_{l'}^v u^3 = \int_{\mathbb{R}} g^v(2^{l'}(x_3-y_3))\partial_3 \Delta_{l'}^v u^3(x_h, y_3)dy_3, \tag{3.4}$$

where $g^v \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{F}(g^v)(\xi_3) = \frac{\check{\varphi}(|\xi_3|)}{i\xi_3}$. Using div$u = 0$, integration by parts, Young's inequality and Lemma 2.6, we get

$$F_j^{3,v}(T) \leq \sum_{\substack{|j-l'|\leq 5 \\ |j-l|\leq 5}} \int_0^T \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}} g^v(2^{l'}(x_3-y_3))\Delta_{l'}^v u^h(x_h, y_3)dy_3 \cdot \nabla_h \partial_3 \Delta_j^v \Delta_l^v a \Delta_j^v a dx \right| dt$$
$$+ \sum_{\substack{|j-l'|\leq 5 \\ |j-l|\leq 5}} \int_0^T \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}} g^v(2^{l'}(x_3-y_3))\Delta_{l'}^v u^h(x_h, y_3)dy_3 \cdot \nabla_h \Delta_j^v a \partial_3 \Delta_j^v \Delta_l^v a dx \right| dt$$

$$\lesssim \sum_{\substack{|j-l'|\leq 5 \\ |j-l|\leq 5}} 2^{l-l'}\|\Delta_{l'}^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))}\|\nabla_h\Delta_j^v a\|_{L_T^2(L^2(\mathbb{R}^3))}\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$

$$\lesssim d_j^2\nu_h^{-\frac{1}{2}-\frac{1}{2p}}2^{-j}\|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}\|a\|^2_{B^{0,\frac{1}{2}}(T)}.$$

Similarly, we have

$$F_j^{4,v}(T)$$

$$\leq \sum_{l\geq j-N_0}\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v\left(\int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v u^h(x_h,y_3)dy_3\cdot\nabla_h\partial_3 S_{l+2}^v a\right)\Delta_j^v a dx\right|dt$$

$$+\sum_{l\geq j-N_0}\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v\left(\int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v u^h(x_h,y_3)dy_3\partial_3 S_{l+2}^v a\right)\cdot\nabla_h\Delta_j^v a dx\right|dt$$

$$\lesssim \sum_{l\geq j-N_0}\|\Delta_l^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))}\|\nabla_h S_{l+2}^v a\|_{L_T^2(L_h^2(L_v^\infty))}\|\Delta_j^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$

$$+\sum_{l\geq j-N_0}\|\Delta_l^v u\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^2))}\|S_{l+2}^v a\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^\infty))}\|\nabla_h\Delta_j^v a\|_{L_T^2(L_h^2(L_v^2))}$$

$$\lesssim d_j^2\nu_h^{-\frac{1}{2}-\frac{1}{2p}}2^{-j}\|u\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}\|a\|^2_{B^{0,\frac{1}{2}}(T)}.$$

This completes the proof of Proposition 3.1. $\qquad\qquad\square$

Proof of Proposition 3.2. We distinguish the terms with horizontal derivatives from the terms with vertical ones, writing

$$G_j(T)\leq G_j^h(T)+G_j^v(T),$$

where

$$G_j^h(T):=\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v(a^h\cdot\nabla_h u_F)\Delta_j^v b dx\right|dt$$

and

$$G_j^v(T):=\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v(a^3\partial_3 u_F)\Delta_j^v b dx\right|dt.$$

Using integration by parts, we have

$$\int_{\mathbb{R}^3}\Delta_j^v(a^h\cdot\nabla_h u_F)\Delta_j^v b dx=-\int_{\mathbb{R}^3}\Delta_j^v(u_F\mathrm{div}_h a^h)\Delta_j^v b dx-\int_{\mathbb{R}^3}\Delta_j^v(a^h\otimes u_F):\nabla_h\Delta_j^v b dx.$$

From Lemmas 2.4 and 3.1-3.2, we have

$$\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v(u_F\mathrm{div}_h a^h)\Delta_j^v b dx\right|dt \leq \|\Delta_j^v(u_F\mathrm{div}_h a^h)\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))}\|\Delta_j^v b\|_{L_T^{2p}(L_h^{\frac{2p}{p-1}}(L_v^2))}$$

$$\lesssim d_j^2\nu_h^{-\frac{1}{2}-\frac{1}{2p}}2^{-j}\|a\|_{B^{0,\frac{1}{2}}(T)}\|b\|_{B^{0,\frac{1}{2}}(T)}\|u_F\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}$$

and

$$\int_0^T\left|\int_{\mathbb{R}^3}\Delta_j^v(a^h\otimes u_F):\nabla_h\Delta_j^v b dx\right|dt$$

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$$\leq \quad \|\Delta_j^v(a^h \otimes u_F)\|_{L_T^2(L^2(\mathbb{R}^3))} \|\Delta_j^v \nabla_h b\|_{L_T^2(L^2(\mathbb{R}^3))}$$

$$\lesssim \quad d_j^2 \nu_h^{-\frac{1}{2}-\frac{1}{2p}} 2^{-j} \|a\|_{B^{0,\frac{1}{2}}(T)} \|b\|_{B^{0,\frac{1}{2}}(T)} \|u_F\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}.$$

On the other hand, using Bony's decomposition in the vertical variables, we obtain

$$\Delta_j^v(a^3 \partial_3 u_F) = \sum_{|j'-j|\leq 5} \Delta_j^v(S_{j'-1}^v a^3 \partial_3 \Delta_{j'}^v u_F) + \sum_{j'\geq j-N_0} \Delta_j^v(\Delta_{j'}^v a^3 \partial_3 S_{j'+2}^v u_F). \qquad (3.5)$$

Using Hölder's inequality, Corollary 2.3, we get

$$\|\Delta_j^v(S_{j'-1}^v a^3 \partial_3 \Delta_{j'}^v u_F)\|_{L_T^1(L^2(\mathbb{R}^3))} \quad \lesssim \quad 2^{j'} \|S_{j'-1}^v a^3\|_{L_T^\infty(L_h^2(L_v^\infty))} \|\Delta_{j'}^v u_F\|_{L_T^1(L_h^\infty(L_v^2))}$$

$$\lesssim \quad d_{j'} 2^{-\frac{j'}{2}} \|u_F\|_{\widetilde{L}_T^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} \|a\|_{B^{0,\frac{1}{2}}(T)}$$

and

$$\sum_{|j'-j|\leq 5} \|\Delta_j^v(S_{j'-1}^v a^3 \partial_3 \Delta_{j'}^v u_F)\|_{L_T^1(L^2(\mathbb{R}^3))} \lesssim d_j 2^{-\frac{j}{2}} \|u_F\|_{\widetilde{L}_T^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} \|a\|_{B^{0,\frac{1}{2}}(T)}.$$

Using Hölder's inequality, Lemmas 2.1, 2.6 and the fact that $\mathrm{div}a = 0$, we have

$$\|\Delta_{j'}^v a^3 S_{j'+2}^v \partial_3 u_F\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))} \quad \lesssim \quad 2^{j'} \|\Delta_{j'}^v a^3\|_{L_T^2(L^2(\mathbb{R}^3))} \|S_{j'+2}^v u_F\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))}$$

$$\lesssim \quad \|\Delta_{j'}^v \partial_3 a^3\|_{L_T^2(L^2(\mathbb{R}^3))} \|S_{j'+2}^v u_F\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))}$$

$$= \quad \|\Delta_{j'}^v \mathrm{div}_h a^h\|_{L_T^2(L^2(\mathbb{R}^3))} \|S_{j'+2}^v u_F\|_{L_T^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))}$$

$$\lesssim \quad d_{j'} \nu_h^{-\frac{1}{2}} 2^{-\frac{j'}{2}} \|u_F\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \|a\|_{B^{0,\frac{1}{2}}(T)}$$

and

$$\sum_{j'\geq j-N_0} \|\Delta_{j'}^v a^3 S_{j'+2}^v \partial_3 u_F\|_{L_T^{\frac{2p}{2p-1}}(L_h^{\frac{2p}{p+1}}(L_v^2))} \lesssim d_j \nu_h^{-\frac{1}{2}} 2^{-\frac{j}{2}} \|u_F\|_{\widetilde{L}_T^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \|a\|_{B^{0,\frac{1}{2}}(T)}.$$

This ends the proof of Proposition 3.2. $\qquad \qquad \square$

4 The proof of the uniqueness

The first step to prove the uniqueness part of Theorem 1.1 is the proof of the following regularity theorem.

Theorem 4.1. *Let* $u \in B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ *be a solution of (1.1) with initial data* $u_0 \in B_p^{-1+\frac{2}{p},\frac{1}{2}}$, $[u_0]_{E_T^p} < \infty$, $p \geq 2$. *Then, there exists a* $T_1 \in (0, T]$ *such that*

$$w = u - u_F \in B^{0,\frac{1}{2}}(T_1).$$

Proof. We already observe at the beginning of Section 3 that the vector field w is the solution of the linear problem, which is

$$\begin{cases} w_t - \nu_h \Delta_h w - \nu_3 \partial_3^2 w = -\nabla P - u \cdot \nabla w - w \cdot \nabla u_F - u_F \cdot \nabla u_F, \\ \mathrm{div}w = 0, \\ w|_{t=0} = u_{0ll}. \end{cases} \qquad (4.1)$$

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Let us apply the operator Δ_j^v to the system (4.1), and set $w_j = \Delta_j^v w$. By the L^2 energy estimate, we have

$$\|w_j(t)\|_{L^2}^2 + 2\nu_h \int_0^t \|\nabla_h w_j(s)\|_{L^2}^2 ds + 2\nu_3 \int_0^t \|\partial_3 w_j(s)\|_{L^2}^2 ds$$

$$\leq \|\Delta_j^v u_{0ll}\|_{L^2}^2 + 2\int_0^t \left| \int_{\mathbb{R}^3} \Delta_j^v(u \cdot \nabla w) w_j dx \right| ds$$

$$+2\int_0^t \left| \int_{\mathbb{R}^3} \Delta_j^v(w \cdot \nabla u_F) w_j dx \right| ds + 2\int_0^t \left| \int_{\mathbb{R}^3} \Delta_j^v(u_F \cdot \nabla u_F) w_j dx \right| ds.$$

From Propositions 3.1-3.2, we obtain, $t \in [0, T]$,

$$\|w_j(t)\|_{L^2}^2 + \nu_h \int_0^t \|\nabla_h w_j(s)\|_{L^2}^2 ds + \nu_3 \int_0^t \|\partial_3 w_j(s)\|_{L^2}^2 ds$$

$$\lesssim \|\Delta_j^v u_{0ll}\|_{L^2}^2 + d_j^2 \nu_h^{-\frac{1}{2}-\frac{1}{2p}} 2^{-j} \|w\|_{B^{0,\frac{1}{2}}(t)}^2 \left(\|u\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} \right)$$

$$+ d_j^2 2^{-j} \|w\|_{B^{0,\frac{1}{2}}(t)}^2 \|u_F\|_{\widetilde{L}_t^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} + d_j^2 2^{-j} \|w\|_{B^{0,\frac{1}{2}}(t)}^2 \|u_F \cdot \nabla u_F\|_{L_t^1(B^{0,\frac{1}{2}})}$$

and

$$\|w\|_{B^{0,\frac{1}{2}}(t)}$$

$$\lesssim \|u_{0ll}\|_{B^{0,\frac{1}{2}}} + \|u_F \cdot \nabla u_F\|_{L_t^1(B^{0,\frac{1}{2}})} + (1 + \nu_h^{-\frac{1}{4}-\frac{1}{4p}}) \|w\|_{B^{0,\frac{1}{2}}(t)}$$

$$\times \left(\|u\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_t^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))} \right)^{\frac{1}{2}}.$$

Thus, we can choose a small $T_1 \in (0, T]$, such that $\|u\|_{\widetilde{L}_{T_1}^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_{T_1}^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))} + \|u_F\|_{\widetilde{L}_{T_1}^1(\widetilde{L}_h^\infty(B_v^{\frac{3}{2}}))}$ is small enough and

$$\|w\|_{B^{0,\frac{1}{2}}(T_1)} \lesssim \|u_{0ll}\|_{B^{0,\frac{1}{2}}} + \|u_F \cdot \nabla u_F\|_{L_{T_1}^1(B^{0,\frac{1}{2}})}.$$

This concludes the proof of Theorem 4.1. $\qquad\square$

The above theorem implies that, if u_i, $i = 1, 2$, are two solutions of (1.1) in the space $B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)$ associated with the same initial data, then there exists a $T_1 \in (0, T]$ such that the difference $\delta := u_2 - u_1$ belongs to $B^{0,\frac{1}{2}}(T_1)$. Moreover, δ satisfies the following system:

$$\begin{cases} \delta_t - \nu_h \Delta_h \delta - \nu_3 \partial_3^2 \delta = L\delta - \nabla P, \\ \operatorname{div}\delta = 0, \\ \delta|_{t=0} = 0, \end{cases} \tag{4.2}$$

where L is the following linear operator

$$L\delta := -\delta \cdot \nabla u_1 - u_2 \cdot \nabla \delta.$$

In order to prove the uniqueness, we have to prove that $\delta \equiv 0$.

As in [3], we give the following definitions.

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Definition 4.1. Let $s \in \mathbb{R}$, and let us define the following semi-norm:

$$\|a\|_{H^{0,s}} := \left(\sum_{j \in \mathbb{Z}} 2^{2js} \|\Delta_j^v a\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Definition 4.2. We denote by $\mathcal{H}$ the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^3)$ by the following norm:

$$\|a\|_{\mathcal{H}}^2 := \sum_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j^{vi} a\|_{L^2(\mathbb{R}^3)}^2 < \infty,$$

where

$$\Delta_j^{vi} = \Delta_j^v, \quad \text{if} \ \ j \geq 0, \quad \text{and} \ \ \Delta_j^{vi} = 0, \quad \text{if} \ \ j \leq -2, \quad \text{and} \ \ \Delta_{-1}^{vi} = S_0^v.$$

Definition 4.3. Let us denote by $\mathcal{B}$ the following semi-norm:

$$\|a\|_{\mathcal{B}}^2 := \sum_{\substack{k \in \mathbb{Z} \\ j \in \mathbb{N}}} 2^{j-k(2-\frac{4}{p})} \|\Delta_k^h \Delta_j^v a\|_{L_h^p(L_v^2)}^2.$$

Remark 4.1. It is obvious that

$$\|a\|_{L_T^\infty(H^{0,\frac{1}{2}})}^2 + \nu_h \|\nabla_h a\|_{L_T^2(H^{0,\frac{1}{2}})}^2 \lesssim \|a\|_{B^{0,\frac{1}{2}}(T)}^2 \tag{4.3}$$

and

$$\|a\|_{L_T^\infty(\mathcal{B})}^2 + \nu_h \|\nabla_h a\|_{L_T^2(\mathcal{B})}^2 \lesssim \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)}^2. \tag{4.4}$$

Let us state the following variation of Lemma 4.2 of [3].

Lemma 4.1. *A constant C exists such that, for any $p' \in [2p, \infty)$, $p \geq 2$, we have*

$$\|\Delta_j^v b\|_{L_h^{p'}(L_v^2)} \leq C c_j \sqrt{p'} 2^{-\frac{j}{2}} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}}, \quad j \geq 0.$$

Proof. From Lemma 2.1, we get

$$
\begin{aligned}
2^{\frac{j}{2}} \|\Delta_j^v b\|_{L_h^{p'}(L_v^2)} \leq \ & C \sum_{k \leq N} 2^{\frac{j}{2}+k(-1+\frac{2}{p})} 2^{k(1-\frac{2}{p'})} \|\Delta_k^h \Delta_j^v b\|_{L_h^p(L_v^2)} \\
& + C \sum_{k \geq N} 2^{\frac{j}{2}+k(-1+\frac{2}{p})} 2^{-\frac{2}{p'}k} \|\Delta_k^h \Delta_j^v \nabla_h b\|_{L_h^p(L_v^2)} \\
\leq \ & C \|b\|_{\mathcal{B}} \sum_{k \leq N} 2^{k(1-\frac{2}{p'})} c_{k,j} + C \|\nabla_h b\|_{\mathcal{B}} \sum_{k \geq N} 2^{-\frac{2}{p'}k} c'_{k,j}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
2^{\frac{j}{2}} \|\Delta_j^v b\|_{L_h^{p'}(L_v^2)} \leq \ & C c_j \|b\|_{\mathcal{B}} \left(\sum_{k \leq N} 2^{2k(1-\frac{2}{p'})} \right)^{\frac{1}{2}} + C c_j \|\nabla_h b\|_{\mathcal{B}} \left(\sum_{k \geq N} 2^{-\frac{4}{p'}k} \right)^{\frac{1}{2}} \\
\leq \ & C c_j \|b\|_{\mathcal{B}} 2^{N(1-\frac{2}{p'})} + C \sqrt{p'} c_j \|\nabla_h b\|_{\mathcal{B}} 2^{-\frac{2}{p'}N}.
\end{aligned}
$$

Choosing $2^N \simeq \frac{\|\nabla_h b\|_{\mathcal{B}}}{\|b\|_{\mathcal{B}}}$ gives the lemma. $\qquad \square$

Let us state the following variation of Lemma 4.1 of [3].

Lemma 4.2. *Let a and b be two divergence free vector fields such that a, $\nabla_h a \in H^{0,\frac{1}{2}} \cap \mathcal{H}$, $b \in \mathcal{B} \cap \widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})$ and $\nabla_h b \in \mathcal{B}$. Let us assume in addition that $\|a\|_{\mathcal{H}}^2 \leq 2^{-2^{2p}}$. Then, we have*

$$|(b \cdot \nabla a|a)_{\mathcal{H}}| + |(a \cdot \nabla b|a)_{\mathcal{H}}| \leq \frac{\nu_h}{10}\|\nabla_h a\|_{\mathcal{H}}^2 + C(a,b)\mu(\|a\|_{\mathcal{H}}^2),$$

where $(f|g)_{\mathcal{H}} := \sum_{j \in \mathbb{Z}} 2^{-j} \int_{\mathbb{R}^3} \Delta_j^{vi} f \Delta_j^{vi} g \, dx$, $\mu(r) := r(1 - \log_2 r)\log_2(1 - \log_2 r)$ and

$$
\begin{aligned}
C(a,b) &:= C_{\nu_h}\big(\|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} + \|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{2p-1}} + \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}^2 \|a\|_{H^{0,\frac{1}{2}}}^{2-\frac{2}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{2}{p}} \\
&\quad + (1 + \|b\|_{\mathcal{B}}^2)\|\nabla_h b\|_{\mathcal{B}}^2\big).
\end{aligned}
$$

Proof. The estimate of the term $(b \cdot \nabla a|a)_{\mathcal{H}}$**.** Using Bony's paradifferential decomposition in the vertical variable and in the inhomogeneous context, we have

$$b \cdot \nabla a = T_b \nabla a + \widetilde{R}(b, \nabla a)$$

with

$$T_b \nabla a := \sum_l S_{l-1}^{vi} b \cdot \nabla \Delta_l^{vi} a, \quad \widetilde{R}(b, \nabla a) := \sum_l \Delta_l^{vi} b \cdot \nabla S_{l+2}^{vi} a \quad \text{and} \quad S_l^{vi} = \sum_{l' \leq l-1} \Delta_{l'}^{vi}.$$

Step 1. The estimate of $(T_b \nabla a|a)_{\mathcal{H}}$. As usual, we shall treat terms involving vertical derivatives in a different way from terms involving horizontal derivatives. This leads to

$$\Delta_j^{vi}(T_b \nabla a) = T_j^h + T_j^v,$$

with

$$T_j^h := \Delta_j^{vi} \sum_{|j-l|\leq 5} S_{l-1}^{vi} b^h \cdot \nabla_h \Delta_l^{vi} a, \quad \text{and} \quad T_j^v := \Delta_j^{vi} \sum_{|j-l|\leq 5} S_{l-1}^{vi} b^3 \partial_3 \Delta_l^{vi} a.$$

Using Hölder's inequality, we obtain

$$\|T_j^h\|_{L_h^{\frac{2p}{p+1}}(L_v^2)} \lesssim \|b\|_{L_h^{2p}(L_v^\infty)} \sum_{|j-l|\leq 5} \|\nabla_h \Delta_l^{vi} a\|_{L^2} \lesssim c_j 2^{\frac{j}{2}} \|b\|_{L_h^{2p}(L_v^\infty)} \|\nabla_h a\|_{\mathcal{H}}$$

and

$$|(T_j^h|\Delta_j^{vi} a)_{L^2}| \lesssim c_j 2^{\frac{j}{2}} \|b\|_{L_h^{2p}(L_v^\infty)} \|\Delta_j^{vi} a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \|\nabla_h a\|_{\mathcal{H}}.$$

Using Minkowski's inequality and the Gagliardo-Nirenberg inequality, we have

$$\|f\|_{L_h^4(L_v^2)} \leq C\|f\|_{L_v^2(L_h^4)} \leq C \left\| \|f(\cdot, x_3)\|_{L_h^2}^{\frac{1}{2}} \|\nabla_h f(\cdot, x_3)\|_{L_h^2}^{\frac{1}{2}} \right\|_{L_v^2} \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\nabla_h f\|_{L^2}^{\frac{1}{2}}.$$

By interpolation, we have

$$\|f\|_{L_h^q(L_v^2)} \leq C\|f\|_{L_h^2(L_v^2)}^{\frac{2}{q}} \|\nabla_h f\|_{L_h^2(L_v^2)}^{1-\frac{2}{q}}, \quad q \in [2,4]. \tag{4.5}$$

Then, we get

$$\|\Delta_j^{vi} a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \lesssim \|\Delta_j^{vi} a\|_{L_h^2(L_v^2)}^{1-\frac{1}{p}} \|\nabla_h \Delta_j^{vi} a\|_{L_h^2(L_v^2)}^{\frac{1}{p}}$$

and

$$\sum_j 2^{-j} |(T_j^h | \Delta_j^{vi} a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)} \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}}. \tag{4.6}$$

The estimate of $(T_j^v | \Delta_j^{vi} a)_{L^2}$ is more delicate. Let us write that

$$T_j^v = \sum_{n=1}^3 T_j^{v,n}$$

with

$$T_j^{v,1} := S_{j-1}^{vi} b^3 \partial_3 \Delta_j^{vi} a, \ T_j^{v,2} := \sum_{|j-l|\le 5} [\Delta_j^{vi}; S_{l-1}^{vi} b^3] \partial_3 \Delta_l^{vi} a,$$

$$T_j^{v,3} := \sum_{|j-l|\le 5} (S_{l-1}^{vi} b^3 - S_{j-1}^{vi} b^3) \partial_3 \Delta_j^{vi} \Delta_l^{vi} a.$$

Step 1a. The estimate of $\sum_j 2^{-j} |(T_j^{v,1} | \Delta_j^{vi} a)_{L^2}|$. To do this, we use the tricks from [3, 5] once again. Using integration by parts and $\mathrm{div} b = 0$, we get

$$\begin{aligned}
(T_j^{v,1} | \Delta_j^{vi} a)_{L^2} &= -\frac{1}{2} \int_{\mathbb{R}^3} S_{j-1}^{vi} \partial_3 b^3 \Delta_j^{vi} a \Delta_j^{vi} a \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} S_{j-1}^{vi} \mathrm{div}_h b^h \Delta_j^{vi} a \Delta_j^{vi} a \, dx \\
&= -\int_{\mathbb{R}^3} S_{j-1}^{vi} b^h \cdot \nabla_h \Delta_j^{vi} a \Delta_j^{vi} a \, dx.
\end{aligned}$$

Similar to the proof of (4.6), we have

$$\sum_j 2^{-j} |(T_j^{v,1} | \Delta_j^{vi} a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)} \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}}. \tag{4.7}$$

Step 1b. The estimate of $\sum_j 2^{-j} |(T_j^{v,2} | \Delta_j^{vi} a)_{L^2}|$. In order to estimate the commutator, let us use the Taylor formula (as in [3]). For a function f on $\mathbb{R}^3$, we define the function $\tilde{f}$ on $\mathbb{R}^4$ by

$$\tilde{f}(x, y_3) := \int_0^1 f(x_h, x_3 + \tau(y_3 - x_3)) d\tau.$$

Then, denoting $\bar{h}(x_3) := x_3 h(x_3)$, we have

$$T_j^{v,2} = \sum_{|j-l|\le 5} \int_{\mathbb{R}} \bar{h}(2^j(x_3 - y_3)) (\widetilde{S_{l-1}^{vi} \partial_3 b^3})(x, y_3) \partial_3 \Delta_l^{vi} a(x_h, y_3) dy_3.$$

Using $\mathrm{div} b = 0$ and $\partial_h \tilde{f} = \widetilde{\partial_h f}$, we obtain

$$T_j^{v,2} = -\sum_{|j-l|\le 5} \int_{\mathbb{R}} \bar{h}(2^j(x_3 - y_3)) \mathrm{div}_h (\widetilde{S_{l-1}^{vi} b^h})(x, y_3) \partial_3 \Delta_l^{vi} a(x_h, y_3) dy_3.$$

Using integration by parts with respect to the horizontal variable, we have

$$(T_j^{v,2} | \Delta_j^{vi} a)_{L^2}$$

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$$= \sum_{|j-l|\leq 5} \left\{ \int_{\mathbb{R}^4} \bar{h}(2^j(x_3-y_3))(\widetilde{S_{l-1}^{vi}b^h})(x,y_3) \cdot \nabla_h \partial_3 \Delta_l^{vi} a(x_h,y_3)\Delta_j^{vi}a(x)dxdy_3 \right.$$
$$\left. + \int_{\mathbb{R}^4} \bar{h}(2^j(x_3-y_3))(\widetilde{S_{l-1}^{vi}b^h})(x,y_3) \cdot \nabla_h \Delta_j^{vi}a(x)\partial_3\Delta_l^{vi}a(x_h,y_3)dxdy_3 \right\}.$$

From $\|\tilde{f}(x_h,\cdot,\cdot)\|_{L^\infty} \leq \|f(x_h,\cdot)\|_{L_v^\infty}$, Young's inequality and Hölder's inequality, we obtain

$$|(T_j^{v,2}|\Delta_j^{vi}a)_{L^2}|$$
$$\lesssim \|b\|_{L_h^{2p}(L_v^\infty)} \sum_{|j-l|\leq 5} \left(\|\Delta_l^{vi}\nabla_h a\|_{L^2}\|\Delta_j^{vi}a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} + \|\Delta_l^{vi}a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)}\|\nabla_h\Delta_j^{vi}a\|_{L^2} \right).$$

Similar to the proof of (4.6), we have

$$\sum_j 2^{-j}|(T_j^{v,2}|\Delta_j^{vi}a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)}\|a\|_{\mathcal{H}}^{1-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}}. \tag{4.8}$$

Step 1c. The estimate of $\sum_j 2^{-j}|(T_j^{v,3}|\Delta_j^{vi}a)_{L^2}|$. For any divergence free vector field u, from (3.4), we have, $l \geq 0$,

$$\Delta_l^v u^3(x) = \int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v \partial_3 u^3(x_h,y_3)dy_3$$
$$= -\mathrm{div}_h \int_{\mathbb{R}} g^v(2^l(x_3-y_3))\Delta_l^v u^h(x_h,y_3)dy_3$$
$$= -2^{-l}\mathrm{div}_h \widetilde{\Delta}_l^v u^h. \tag{4.9}$$

If $j \geq 7$, then the terms $S_{l-1}^{vi}b^3 - S_{j-1}^{vi}b^3$ that appear in $T_j^{v,3}$ are a sum of the terms $\Delta_{l'}^{vi}$ with $l' \geq 0$. If $j \geq 7$, using (4.9) and integration by parts in the horizontal variable, we obtain

$$(T_j^{v,3}|\Delta_j^{vi}a)_{L^2} = \sum_{\substack{|l'-l|\leq 5 \\ |l-j|\leq 5}} 2^{-l'} \left(\int_{\mathbb{R}^3} \Delta_j^{vi}(\widetilde{\Delta}_{l'}^{vi}b^h \cdot \nabla_h \Delta_l^{vi}\partial_3 a)\Delta_j^{vi}adx \right.$$
$$\left. + \int_{\mathbb{R}^3} \Delta_j^{vi}(\Delta_l^{vi}\partial_3 a \widetilde{\Delta}_{l'}^{vi}b^h) \cdot \nabla_h \Delta_j^{vi}adx \right).$$

Similar to the proof of (4.8), we have

$$\sum_{j\geq 7} 2^{-j}|(T_j^{v,3}|\Delta_j^{vi}a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)}\|a\|_{\mathcal{H}}^{1-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}}. \tag{4.10}$$

If $j \leq 7$, we can easily get

$$|(T_j^{v,3}|\Delta_j^{vi}a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)}\|a\|_{\mathcal{H}}^{2-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}}.$$

Plugging this inequality with inequalities (4.6)-(4.8) and (4.10), using Young's inequality, we have

$$|(T_b\nabla a|a)_{\mathcal{H}}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)} \left(\|a\|_{\mathcal{H}}^{1-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}} + \|a\|_{\mathcal{H}}^{2-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} \right)$$

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$$\leq \frac{\nu_h}{100}\|\nabla_h a\|_{\mathcal{H}}^2 + C_{\nu_h}\|a\|_{\mathcal{H}}^2 \left(\|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} + \|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{2p-1}}\right). \qquad (4.11)$$

Step 2. The estimate of $(\widetilde{R}(b,\nabla a)|a)_{\mathcal{H}}$. Again, let us treat terms involving vertical derivatives in a different way from terms involving horizontal derivatives. This leads to

$$\Delta_j^{vi}\widetilde{R}(b,\nabla a) = R_j^h + R_j^v + R_j^0$$

with

$$R_j^h := \Delta_j^{vi} \sum_{l\geq(j-N_0)^+} \Delta_l^v b^h \cdot \nabla_h S_{l+2}^v a,$$

$$R_j^v := \Delta_j^{vi} \sum_{l\geq(j-N_0)^+} \Delta_l^v b^3 \partial_3 S_{l+2}^v a,$$

$$R_j^0 := \Delta_j^{vi}(S_0^v b \cdot \nabla S_2^v a).$$

Let us first estimate R_j^0. It is obvious that if j is large enough, this term is 0. Thus, if $j \leq N_1$, we obtain

$$\sum_{j\leq N_1} |(R_j^0|\Delta_j^{vi}a)_{L^2}| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)}\left(\|a\|_{\mathcal{H}}^{1-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}} + \|a\|_{\mathcal{H}}^{2-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}}\right)$$

$$\leq \frac{\nu_h}{100}\|\nabla_h a\|_{\mathcal{H}}^2 + C_{\nu_h}\|a\|_{\mathcal{H}}^2\left(\|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} + \|b\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{2p-1}}\right). \qquad (4.12)$$

Step 2a. The estimate of $\sum_j 2^{-j}|(R_j^h|\Delta_j^{vi}a)_{L^2}|$. First, we estimate R_j^h in high (vertical) frequencies. From Lemma 2.1 and Hölder's inequality, we have

$$\|R_j^h\|_{L_h^{\frac{2p}{p+1}}(L_v^2)} \lesssim 2^{\frac{j}{2}}\sum_{l\geq(j-N_0)^+}\|\Delta_l^v b^h \cdot \nabla_h S_{l+2}^v a\|_{L_h^{\frac{2p}{p+1}}(L_v^1)}$$

$$\lesssim 2^{\frac{j}{2}}\sum_{l\geq(j-N_0)^+}\|\Delta_l^v b^h\|_{L_h^{2p}(L_v^2)}\|\nabla_h S_{l+2}^v a\|_{L_h^2(L_v^2)}$$

$$\lesssim 2^{\frac{j}{2}}\sum_{l\geq(j-N_0)^+}\|\Delta_l^v b^h\|_{L_h^{2p}(L_v^2)}\sum_{l'=-1}^{l+1}\|\nabla_h \Delta_{l'}^{vi}a\|_{L_h^2(L_v^2)}$$

$$\lesssim \|b^h\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}\|\nabla_h a\|_{\mathcal{H}}\sum_{l\geq(j-N_0)^+}d_l 2^{\frac{1}{2}(j-l)}\left(\sum_{l'=-1}^{l+1}2^{l'}\right)^{\frac{1}{2}}$$

$$\lesssim 2^{\frac{j}{2}}\|b^h\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}\|\nabla_h a\|_{\mathcal{H}}.$$

Combining it with (4.5), we have

$$|(R_j^h|\Delta_j^{vi}a)_{L^2}| \lesssim 2^{\frac{j}{2}}\|b^h\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}\|\nabla_h a\|_{\mathcal{H}}\|\Delta_j^{vi}a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)}$$

$$\lesssim \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}\|\nabla_h a\|_{\mathcal{H}}\|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}}\|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}. \qquad (4.13)$$

Then, we estimate R_j^h in low (vertical) frequencies. Following the idea of [3, 12], using Lemmas 2.1 and 4.1, we obtain

$$\|R_j^h\|_{L_h^{\frac{2p'}{p'+2}}(L_v^2)} \lesssim 2^{\frac{j}{2}}\sum_{l\geq(j-N_0)^+}\|\Delta_l^v b^h \cdot \nabla_h S_{l+2}^v a\|_{L_h^{\frac{2p'}{p'+2}}(L_v^1)}$$

$$\lesssim 2^{\frac{j}{2}} \sum_{l \geq (j-N_0)^+} \|\Delta_l^v b^h\|_{L_h^{p'}(L_v^2)} \|\nabla_h S_{l+2}^v a\|_{L_h^2(L_v^2)}$$

$$\lesssim 2^{\frac{j}{2}} \sum_{l \geq (j-N_0)^+} c_l 2^{-\frac{l}{2}} \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \sum_{k \leq l+1} c_k 2^{\frac{k}{2}} \|\nabla_h a\|_{\mathcal{H}}$$

$$\lesssim 2^{\frac{j}{2}} \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}.$$

By (4.5), a constant C exists (independent of p') such that,

$$\|\Delta_j^{vi} a\|_{L_h^{\frac{2p'}{p'-2}}(L_v^2)} \leq C \|\Delta_j^{vi} a\|_{L^2}^{1-\frac{2}{p'}} \|\nabla_h \Delta_j^{vi} a\|_{L^2}^{\frac{2}{p'}}.$$

Thus we get

$$|(R_j^h | \Delta_j^{vi} a)_{L^2}| \lesssim c_j 2^j \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{2}{p'}}. \qquad (4.14)$$

From (4.13)-(4.14), the Cauchy-Schwarz inequality and Young's inequality, we have

$$\sum_j 2^{-j} |(R_j^h | \Delta_j^{vi} a)_{L^2}| \lesssim \sum_{j>M} 2^{-j} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} \|\nabla_h a\|_{\mathcal{H}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}$$

$$+ \sum_{-1 \leq j \leq M} c_j \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{2}{p'}}$$

$$\lesssim 2^{-M} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} \|\nabla_h a\|_{\mathcal{H}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}$$

$$+ \sqrt{p'M} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{2}{p'}}$$

$$\leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + \frac{C}{\nu_h} 2^{-2M} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}^2 \|a\|_{H^{0,\frac{1}{2}}}^{2-\frac{2}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{2}{p}}$$

$$+ C \nu_h^{-\frac{p'+2}{p'-2}} (p'M)^{\frac{p'}{p'-2}} \|b\|_{\mathcal{B}}^{\frac{4}{p'-2}} \|\nabla_h b\|_{\mathcal{B}}^2 \|a\|_{\mathcal{H}}^2.$$

Let us assume that $M \geq 2^{2p}$. Choosing $p' = \log_2 M$, we get

$$\sum_j 2^{-j} |(R_j^h | \Delta_j^{vi} a)_{L^2}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + \frac{C}{\nu_h} 2^{-2M} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}^2 \|a\|_{H^{0,\frac{1}{2}}}^{2-\frac{2}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{2}{p}}$$

$$+ C_{\nu_h} (1 + \|b\|_{\mathcal{B}}^2) \|\nabla_h b\|_{\mathcal{B}}^2 \|a\|_{\mathcal{H}}^2 M \log_2 M.$$

If $\|a\|_{\mathcal{H}} \leq 2^{-2^{2p}}$, then we can choose M such that $2^{-M} \simeq \|a\|_{\mathcal{H}}$, and get

$$\sum_j 2^{-j} |(R_j^h | \Delta_j^{vi} a)_{L^2}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a, b) \mu(\|a\|_{\mathcal{H}}), \qquad (4.15)$$

with

$$C_1(a, b) = \frac{C}{\nu_h} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}^2 \|a\|_{H^{0,\frac{1}{2}}}^{2-\frac{2}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{2}{p}} + C_{\nu_h} (1 + \|b\|_{\mathcal{B}}^2) \|\nabla_h b\|_{\mathcal{B}}^2.$$

Step 2b. The estimate of $\sum_j 2^{-j} |(R_j^v | \Delta_j^{vi} a)_{L^2}|$. From (4.9) and integration by parts in the horizontal variable, we have

$$(R_j^v | \Delta_j^{vi} a)_{L^2} = R_j^{v,1} + R_j^{v,2}$$

with

$$R_j^{v,1} := \sum_{l \geq (j-N_0)^+} 2^{-l} \int_{\mathbb{R}^3} \Delta_j^{vi} (\widetilde{\Delta}_l^v b^h \cdot \nabla_h \partial_3 S_{l+2}^v a) \Delta_j^{vi} a\, dx,$$

$$R_j^{v,2} := \sum_{l \geq (j-N_0)^+} 2^{-l} \int_{\mathbb{R}^3} \Delta_j^{vi} (\widetilde{\Delta}_l^v b^h \partial_3 S_{l+2}^v a) \cdot \nabla_h \Delta_j^{vi} a\, dx.$$

Since $a \in H^{0,\frac{1}{2}} \cap \mathcal{H}$, we have

$$\|\partial_3 S_l^v a\|_{L^2} \lesssim c_l 2^{\frac{3l}{2}} \|a\|_{\mathcal{H}} \quad \text{and} \quad \|\partial_3 S_l^v a\|_{L^2} \lesssim c_l 2^{\frac{l}{2}} \|a\|_{H^{0,\frac{1}{2}}}, \tag{4.16}$$

using the similar argument to that in the proof of (4.15), we have

$$\sum_j 2^{-j} |R_j^{v,1}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}^2). \tag{4.17}$$

Now let us estimate $R_j^{v,2}$ in high (vertical) frequencies by using that a and $\nabla_h a$ are in $H^{0,\frac{1}{2}}$. Using Lemma 2.1, we obtain

$$\|\Delta_j^{vi}(\widetilde{\Delta}_l^v b^h \partial_3 S_{l+2}^v a)\|_{L^2} \lesssim 2^{\frac{j}{2}} \|\widetilde{\Delta}_l^v b^h \partial_3 S_{l+2}^v a\|_{L_h^2(L_v^1)}$$

$$\lesssim 2^{\frac{j}{2}} \|\widetilde{\Delta}_l^v b^h\|_{L_h^{2p}(L_v^2)} \|\partial_3 S_{l+2}^v a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)}.$$

Using (4.5) and (4.16), we obtain

$$\|\partial_3 S_{l+2}^v a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \lesssim c_l 2^{\frac{l}{2}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}}$$

and

$$|R_j^{v,2}| \lesssim \|b^h\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} \|\nabla_h a\|_{\mathcal{H}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}. \tag{4.18}$$

Then, let us estimate $R_j^{v,2}$ in low (vertical) frequencies by using that a and $\nabla_h a$ are in $\mathcal{H}$. Using Lemma 4.1, (4.5) and (4.16), we have

$$\|\Delta_j^{vi}(\widetilde{\Delta}_l^v b^h \partial_3 S_{l+2}^v a)\|_{L^2} \lesssim 2^{\frac{j}{2}} \|\widetilde{\Delta}_l^v b\|_{L_h^{p'}(L_v^2)} \|\partial_3 S_{l+2}^v a\|_{L_h^{\frac{2p'}{p'-2}}(L_v^2)}$$

$$\lesssim 2^{\frac{j}{2}} 2^l d_l \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{2}{p'}}.$$

Thus, we deduce that

$$|R_j^{v,2}| \lesssim c_j 2^j \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{2}{p'}}.$$

Combining it with (4.18), using the similar argument to that in the proof of (4.15), we have

$$\sum_j 2^{-j} |(R_j^h | \Delta_j^{vi} a)_{L^2}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}). \tag{4.19}$$

This proves the estimate of the term $(b \cdot \nabla a | a)_{\mathcal{H}}$.

The estimate of the term $(a \cdot \nabla b | a)_{\mathcal{H}}$. Using Bony's paradifferential decomposition in the vertical variable and in the inhomogeneous context, we have

$$a \cdot \nabla b = T_a \nabla b + R(a, \nabla b) + T_{\nabla b} a$$

with

$$T_a \nabla b := \sum_{l \geq 1} S_{l-1}^{vi} a \cdot \nabla \Delta_l^{vi} b, \ \ R(a, \nabla b) := \sum_{\substack{l \geq -1 \\ i = -1, 0, 1}} \Delta_l^{vi} a \cdot \nabla \Delta_{l+i}^{vi} b$$

$$\text{and} \ \ T_{\nabla b} a := \sum_{l \geq 1} \Delta_l^{vi} a \cdot \nabla S_{l-1}^{vi} b.$$

Step 3. The estimate of $(T_a \nabla b | a)_{\mathcal{H}}$. Using the similar argument to that in Step 1, we have

$$\Delta_j^{vi}(T_a \nabla b) = \overline{T}_j^h + \overline{T}_j^v,$$

with

$$\overline{T}_j^h := \Delta_j^{vi} \sum_{|j-l| \leq 5} S_{l-1}^{vi} a^h \cdot \nabla_h \Delta_l^{vi} b \ \ \text{and} \ \ \overline{T}_j^v := \Delta_j^{vi} \sum_{|j-l| \leq 5} S_{l-1}^{vi} a^3 \partial_3 \Delta_l^{vi} b.$$

Step 3a. The estimate of $\sum_j 2^{-j}(\overline{T}_j^h | \Delta_j^{vi} a)_{L^2}$. Then, we have

$$(\overline{T}_j^h | \Delta_j^{vi} a)_{L^2} = \overline{T}_j^{h1} + \overline{T}_j^{h2},$$

with

$$\overline{T}_j^{h1} := -\int_{\mathbb{R}^3} \Delta_j^{vi} \Big(\sum_{|j-l| \leq 5} S_{l-1}^{vi} \text{div}_h a^h \Delta_l^{vi} b \Big) \Delta_j^{vi} a \, dx,$$

$$\overline{T}_j^{h2} := -\int_{\mathbb{R}^3} \Delta_j^{vi} \Big(\sum_{|j-l| \leq 5} S_{l-1}^{vi} a^h \Delta_l^{vi} b \Big) \cdot \nabla_h \Delta_j^{vi} a \, dx.$$

Using Hölder's inequality and (4.5), we obtain

$$\begin{aligned} |\overline{T}_j^{h1}| &\lesssim \sum_{|j-l| \leq 5} \|\nabla_h S_{l-1}^{vi} a\|_{L^2} \|b\|_{L_h^{2p}(L_v^\infty)} \|\Delta_j^{vi} a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \\ &\lesssim \sum_{|j-l| \leq 5} c_l 2^{\frac{l}{2}} \|\nabla_h a\|_{\mathcal{H}} \|b\|_{L_h^{2p}(L_v^\infty)} c_j 2^{\frac{j}{2}} \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} \\ &\lesssim d_j 2^j \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}} \|b\|_{L_h^{2p}(L_v^\infty)}, \end{aligned}$$

$$\begin{aligned} |\overline{T}_j^{h2}| &\lesssim \sum_{|j-l| \leq 5} \|S_{l-1}^{vi} a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \|b\|_{L_h^{2p}(L_v^\infty)} \|\nabla_h \Delta_j^{vi} a\|_{L^2} \\ &\lesssim \sum_{|j-l| \leq 5} c_l 2^{\frac{l}{2}} \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} \|b\|_{L_h^{2p}(L_v^\infty)} c_j 2^{\frac{j}{2}} \|\nabla_h a\|_{\mathcal{H}} \\ &\lesssim d_j 2^j \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}} \|b\|_{L_h^{2p}(L_v^\infty)}. \end{aligned}$$

Step 3b. The estimate of $\sum_j 2^{-j}(\overline{T}_j^v | \Delta_j^{vi} a)_{L^2}$. Let

$$\overline{T}_j^v := \overline{T}_j^{v0} + \overline{T}_j^{v1},$$

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with

$$\overline{T}_j^{v0} := \Delta_j^{vi} \sum_{|j-l|\leq 5} S_0^{vi} a^3 \partial_3 \Delta_l^{vi} b,$$

$$\overline{T}_j^{v1} := \Delta_j^{vi} \sum_{\substack{|j-l|\leq 5 \\ l'\in[0,l-2]}} \Delta_{l'}^{vi} a^3 \partial_3 \Delta_l^{vi} b.$$

Using Hölder's inequality and (4.5), we obtain

$$
\begin{aligned}
|(\overline{T}_j^{v0}|\Delta_j^{vi}a)_{L^2}| &\lesssim \sum_{|j-l|\leq 5} \|S_0^{vi}a\|_{L_h^2(L_v^\infty)} \|\partial_3\Delta_l^{vi}b\|_{L_h^{2p}(L_v^2)} \|\Delta_j^{vi}a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \\
&\lesssim \sum_{|j-l|\leq 5} \|a\|_{\mathcal{H}} d_l 2^{\frac{l}{2}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} c_j 2^{\frac{j}{2}} \|a\|_{\mathcal{H}}^{1-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} \\
&\lesssim d_j 2^j \|a\|_{\mathcal{H}}^{2-\frac{1}{p}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})}.
\end{aligned}
$$

From (4.9), we have

$$|\sum_j 2^{-j}(\overline{T}_j^{v1}|\Delta_j^{vi}a)_{L^2}| \leq \overline{T}^{v1,N} + \overline{T}_N^{v1},$$

with

$$\overline{T}^{v1,N} = \sum_{j\geq N} \sum_{\substack{|j-l|\leq 5 \\ l'\geq N}} 2^{-j} \left| \int_{\mathbb{R}^3} \Delta_j^{vi}(2^{-l'}\widetilde{\Delta}_{l'}^{vi}\mathrm{div}_h a^h \partial_3 \Delta_l^{vi} b)\Delta_j^{vi} a\, dx \right|,$$

$$\overline{T}_N^{v1} = \sum_j \sum_{\substack{|j-l|\leq 5 \\ l'\in[0,N]}} 2^{-j} \left| \int_{\mathbb{R}^3} \Delta_j^{vi}(2^{-l'}\widetilde{\Delta}_{l'}^{vi}\mathrm{div}_h a^h \partial_3 \Delta_l^{vi} b)\Delta_j^{vi} a\, dx \right|.$$

From (4.5), Lemma 4.1 and Hölder's inequality, we obtain

$$
\begin{aligned}
\overline{T}_N^{v1} &\lesssim \sum_j 2^{-j} \sum_{\substack{|j-l|\leq 5 \\ l'\in[0,N]}} 2^{-l'} \|\nabla_h \Delta_{l'}^{vi}a\|_{L_h^2(L_v^\infty)} \|\partial_3\Delta_l^{vi}b\|_{L_h^{p'}(L_v^2)} \|\Delta_j^{vi}a\|_{L_h^{\frac{2p'}{p'-2}}(L_v^2)} \\
&\lesssim \sum_j 2^{-j} \sum_{\substack{|j-l|\leq 5 \\ l'\in[0,N]}} c_{l'} \|\nabla_h a\|_{\mathcal{H}} c_l 2^{\frac{l}{2}} \sqrt{p'} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} c_j 2^{\frac{j}{2}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{\frac{2}{p'}} \\
&\lesssim \sqrt{p'N} \|b\|_{\mathcal{B}}^{\frac{2}{p'}} \|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} \|a\|_{\mathcal{H}}^{1-\frac{2}{p'}} \|\nabla_h a\|_{\mathcal{H}}^{1+\frac{2}{p'}}, \quad \text{(low vertical frequencies)}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{T}^{v1,N} &\lesssim \sum_{j\geq N} 2^{-\frac{j}{2}} \sum_{\substack{|j-l|\leq 5 \\ l'\geq N}} 2^{-l'} \|\nabla_h \Delta_{l'}^{vi}a\|_{L_h^2(L_v^2)} \|\partial_3\Delta_l^{vi}b\|_{L_h^{2p}(L_v^2)} \|\Delta_j^{vi}a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \\
&\lesssim \sum_{j\geq N} 2^{-\frac{j}{2}} \sum_{\substack{|j-l|\leq 5 \\ l'\geq N}} 2^{-\frac{l'}{2}} c_{l'} \|\nabla_h a\|_{\mathcal{H}} d_l 2^{\frac{l}{2}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} 2^{-\frac{j}{2}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}} \\
&\lesssim 2^{-N} \|\nabla_h a\|_{\mathcal{H}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}, \quad \text{(high vertical frequencies)}.
\end{aligned}
$$

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Using the similar argument to that in the proof of (4.15), we get

$$|\sum_j 2^{-j}(\overline{T}_j^{v1}|\Delta_j^{vi}a)_{L^2}| \le \frac{\nu_h}{200}\|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}).$$

Thus, from above estimates, we get

$$(T_a\nabla b|a)_{\mathcal{H}} \le \frac{\nu_h}{100}\|\nabla_h a\|_{\mathcal{H}}^2 + C(a,b)\mu(\|a\|_{\mathcal{H}}^2).$$

Step 4. The estimate of $(R(a,\nabla b)|a)_{\mathcal{H}}$. Obviously, we have

$$(\Delta_j^{vi} R(a,\nabla b)|\Delta_j^{vi}a)_{L^2} = \overline{R}_j^h + \overline{R}_j^v + \overline{R}_j^0,$$

with

$$\overline{R}_j^h := (\Delta_j^{vi}(\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} \Delta_l^{vi} a^h \cdot \nabla_h \Delta_{l+i}^{vi} b)|\Delta_j^{vi}a)_{L^2},$$

$$\overline{R}_j^v := (\Delta_j^{vi}(\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} \Delta_l^{vi} a^3 \partial_3 \Delta_{l+i}^{vi} b)|\Delta_j^{vi}a)_{L^2},$$

$$\overline{R}_j^0 := (\Delta_j^{vi}(S_0^{vi} a \cdot \nabla S_1^{vi} b)|\Delta_j^{vi}a)_{L^2} + (\Delta_j^{vi}(\Delta_0^{vi} a \cdot \nabla S_2^{vi} b)|\Delta_j^{vi}a)_{L^2}.$$

It is obvious that if j is large enough, $\overline{R}_j^0$ is 0. Thus, if $j \le N_1$, we obtain

$$|\overline{R}_j^0| \lesssim \|b\|_{L_h^{2p}(L_v^\infty)}\left(\|a\|_{\mathcal{H}}^{2-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{\frac{1}{p}} + \|a\|_{\mathcal{H}}^{1-\frac{1}{p}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p}}\right).$$

Step 4a. The estimate of $\sum_j 2^{-j}\overline{R}_j^h$. Then, we have

$$\overline{R}_j^h = \overline{R}_j^{h1} + \overline{R}_j^{h2},$$

with

$$\overline{R}_j^{h1} := -\int_{\mathbb{R}^3}\Delta_j^{vi}(\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} \Delta_l^{vi}\text{div}_h a^h \Delta_{l+i}^{vi} b)\Delta_j^{vi}a\,dx,$$

$$\overline{R}_j^{h2} := -\int_{\mathbb{R}^3}\Delta_j^{vi}(\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} \Delta_l^{vi} a^h \Delta_{l+i}^{vi} b) \cdot \nabla_h \Delta_j^{vi}a\,dx.$$

Using Hölder's inequality and (4.5), we obtain

$$\begin{aligned}|\overline{R}_j^{h1}| &\lesssim 2^{\frac{j}{2}}\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} \|\nabla_h \Delta_l^{vi} a\|_{L^2}\|\Delta_{l+i}^{vi} b\|_{L_h^{p'}(L_v^2)}\|\Delta_j^{vi}a\|_{L_h^{\frac{2p'}{p'-2}}(L_v^2)}\\ &\lesssim 2^{\frac{j}{2}}\sum_{\substack{l \ge 1+(j-N_0)^+ \\ i=-1,0,1}} c_l 2^{\frac{l}{2}}\|\nabla_h a\|_{\mathcal{H}} c_{l+i} 2^{-\frac{l}{2}}\sqrt{p'}\|b\|_{\mathcal{B}}^{\frac{2}{p'}}\|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}} c_j 2^{\frac{j}{2}}\|a\|_{\mathcal{H}}^{1-\frac{2}{p'}}\|\nabla_h a\|_{\mathcal{H}}^{\frac{2}{p'}}\\ &\lesssim c_j\sqrt{p'}2^j\|a\|_{\mathcal{H}}^{1-\frac{1}{p'}}\|\nabla_h a\|_{\mathcal{H}}^{1+\frac{1}{p'}}\|b\|_{\mathcal{B}}^{\frac{2}{p'}}\|\nabla_h b\|_{\mathcal{B}}^{1-\frac{2}{p'}}, \text{ (low vertical frequencies)},\end{aligned}$$

$$|\overline{R}_j^{h1}| \lesssim 2^{\frac{j}{2}} \sum_{\substack{l \geq 1+(j-N_0)^+ \\ i=-1,0,1}} \|\nabla_h \Delta_l^{vi} a\|_{L^2} \|\Delta_{l+i}^{vi} b\|_{L_h^{2p}(L_v^2)} \|\Delta_j^{vi} a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)}$$

$$\lesssim 2^{\frac{j}{2}} \sum_{\substack{l \geq 1+(j-N_0)^+ \\ i=-1,0,1}} c_l 2^{\frac{l}{2}} \|\nabla_h a\|_{\mathcal{H}} d_{l+i} 2^{-\frac{l}{2}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} c_j 2^{-\frac{j}{2}} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}$$

$$\lesssim \|\nabla_h a\|_{\mathcal{H}} \|b\|_{\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}})} \|a\|_{H^{0,\frac{1}{2}}}^{1-\frac{1}{p}} \|\nabla_h a\|_{H^{0,\frac{1}{2}}}^{\frac{1}{p}}. \text{ (high vertical frequencies)}.$$

Using the similar argument to that in the proof of (4.15), we get

$$|\overline{R}_j^{h1}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}). \tag{4.20}$$

Similarly, we have

$$|\overline{R}_j^{h2}| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}).$$

From (4.9), we have

$$\overline{R}_j^v = -(\Delta_j^{vi}(\sum_{\substack{l \geq 1+(j-N_0)^+ \\ i=-1,0,1}} 2^{-l} \widetilde{\Delta}_l^{vi} \mathrm{div}_h a^h \partial_3 \Delta_{l+i}^{vi} b) | \Delta_j^{vi} a)_{L^2},$$

and using the similar argument to that in the proof of (4.20), we obtain

$$|\overline{R}_j^v| \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C_1(a,b)\mu(\|a\|_{\mathcal{H}}).$$

Step 5. The estimate of $(T_{\nabla b} a | a)_{\mathcal{H}}$. Using similar arguments to that in the proof of Steps 3-4, we get

$$(T_{\nabla b} a | a)_{\mathcal{H}} \leq \frac{\nu_h}{100} \|\nabla_h a\|_{\mathcal{H}}^2 + C(a,b)\mu(\|a\|_{\mathcal{H}}^2).$$

This proves Lemma 4.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Then, we will prove that $\delta \in L_{T_1}^\infty(\mathcal{H})$ and $\nabla_h \delta \in L_{T_1}^2(\mathcal{H})$ in the following lemma.

Lemma 4.3. *We have* $\delta \in L_{T_1}^\infty(\mathcal{H})$ *and* $\nabla_h \delta \in L_{T_1}^2(\mathcal{H})$.

Proof. Since $\delta \in B^{0,\frac{1}{2}}$, we only need to prove that $S_0^v \delta \in L_{T_1}^\infty(L^2)$ and $\nabla_h S_0^v \delta \in L_{T_1}^2(L^2)$.
Let us write that $S_0^v \delta$ is a solution (with initial value 0) of

$$\partial_t S_0^v \delta - \nu_h \Delta_h S_0^v \delta - \nu_3 \partial_3^2 S_0^v \delta = \sum_{i=1}^6 g_i - \nabla S_0^v P,$$

with

$$g_1 := -S_0^v \partial_3 (\delta^3 u_1),$$
$$g_2 := -S_0^v \mathrm{div}_h (u_1 (Id - S_0^v) \delta^h),$$
$$g_3 := -S_0^v \mathrm{div}_h (u_1 S_0^v \delta^h),$$
$$g_4 := -S_0^v \partial_3 (u_2^3 \delta),$$
$$g_5 := -S_0^v \mathrm{div}_h (u_2^h (Id - S_0^v) \delta),$$
$$g_6 := -S_0^v \mathrm{div}_h (u_2^h S_0^v \delta).$$

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Using Lemmas 2.1 and 3.2, we have

$$
\begin{aligned}
\|g_1\|_{L^2_{T_1}(L^2)} &\lesssim \sum_{j \leq -1} 2^j \|\Delta^v_j(u_1 \delta^3)\|_{L^2_{T_1}(L^2(\mathbb{R}^3))} \\
&\lesssim \sum_{j \leq -1} 2^j d_j \nu_h^{-\frac{1}{2p}} 2^{-\frac{j}{2}} \|\delta\|_{B^{0,\frac{1}{2}}(T_1)} \|u_1\|_{\widetilde{L}^{\frac{2p}{p-1}}_{T_1}(\widetilde{L}^{2p}_h(B^{\frac{1}{2}}_v))} \\
&\lesssim \nu_h^{-\frac{1}{2p}} \|\delta\|_{B^{0,\frac{1}{2}}(T_1)} \|u_1\|_{\widetilde{L}^{\frac{2p}{p-1}}_{T_1}(\widetilde{L}^{2p}_h(B^{\frac{1}{2}}_v))}.
\end{aligned}
\tag{4.21}
$$

Similarly, we have

$$
\|g_4\|_{L^2_{T_1}(L^2)} \lesssim \nu_h^{-\frac{1}{2p}} \|\delta\|_{B^{0,\frac{1}{2}}(T_1)} \|u_2\|_{\widetilde{L}^{\frac{2p}{p-1}}_{T_1}(\widetilde{L}^{2p}_h(B^{\frac{1}{2}}_v))}.
\tag{4.22}
$$

By (4.5) and Young's inequality, we obtain

$$
\begin{aligned}
\|(Id - S^v_0)\delta\|_{L^{2p}_{T_1}(L^{\frac{2p}{p-1}}_h(L^2_v))} &\lesssim \sum_{j \geq 0} \|\Delta^v_j \delta\|_{L^{2p}_{T_1}(L^{\frac{2p}{p-1}}_h(L^2_v))} \\
&\lesssim \sum_{j \geq 0} \|\Delta^v_j \delta\|^{1-\frac{1}{p}}_{L^\infty_{T_1}(L^2_h(L^2_v))} \|\Delta^v_j \nabla_h \delta\|^{\frac{1}{p}}_{L^2_{T_1}(L^2_h(L^2_v))} \\
&\lesssim \nu_h^{-\frac{1}{2p}} \sum_{j \geq 0} \left(\|\Delta^v_j \delta\|_{L^\infty_{T_1}(L^2_h(L^2_v))} + \nu_h^{\frac{1}{2}} \|\Delta^v_j \nabla_h \delta\|_{L^2_{T_1}(L^2_h(L^2_v))} \right) \\
&\lesssim \nu_h^{-\frac{1}{2p}} \|\delta\|_{B^{0,\frac{1}{2}}(T_1)}.
\end{aligned}
$$

Then, we have

$$
g_2 = \text{div}_h \widetilde{g_2} \quad \text{and} \quad g_5 = \text{div}_h \widetilde{g_5},
\tag{4.23}
$$

with

$$
\|\widetilde{g_2}\|_{L^2_t(L^2)} \lesssim \nu_h^{-\frac{1}{2p}} \|\delta\|_{B^{0,\frac{1}{2}}(t)} \|u_1\|_{\widetilde{L}^{\frac{2p}{p-1}}_t(\widetilde{L}^{2p}_h(B^{\frac{1}{2}}_v))},
$$

$$
\|\widetilde{g_5}\|_{L^2_t(L^2)} \lesssim \nu_h^{-\frac{1}{2p}} \|\delta\|_{B^{0,\frac{1}{2}}(t)} \|u_2\|_{\widetilde{L}^{\frac{2p}{p-1}}_t(\widetilde{L}^{2p}_h(B^{\frac{1}{2}}_v))}.
$$

The terms g_3 and g_6 must be treated with a commutator argument based on the following estimate, which is proved in Lemma 4.3 of [3]: Let χ be a function of $\mathcal{S}(\mathbb{R})$. A constant C exists such that, for any function a in $L^2_h(L^\infty_v)$, we have

$$
\|[\chi(\varepsilon x_3); S^v_0]a\|_{L^2} \leq C \varepsilon^{\frac{1}{2}} \|a\|_{L^2_h(L^\infty_v)}.
\tag{4.24}
$$

Now let us choose $\chi \in \mathcal{D}(\mathbb{R})$ with value 1 near 0 and let us state

$$
S^v_{0,\varepsilon} a := \chi(\varepsilon x_3) S^v_0 a.
$$

Using a classical L^2 energy estimate and Young's inequality, we have

$$
\begin{aligned}
&\|S^v_{0,\varepsilon} \delta(t)\|^2_{L^2} + \nu_h \int_0^t \|\nabla_h S^v_{0,\varepsilon} \delta(s)\|^2_{L^2} ds + 2\nu_3 \int_0^t \|\partial_3 S^v_{0,\varepsilon} \delta(s)\|^2_{L^2} ds \\
&\lesssim \int_0^t \|S^v_{0,\varepsilon} \delta(s)\|_{L^2} \left(\|g_1(s)\|_{L^2} + \|g_4(s)\|_{L^2} \right) ds + \frac{1}{\nu_h} \int_0^t \left(\|\widetilde{g_2}(s)\|^2_{L^2} + \|\widetilde{g_5}(s)\|^2_{L^2} \right) ds
\end{aligned}
$$

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$$+ \int_0^t \int_{\mathbb{R}^3} \chi(\varepsilon x_3)(g_3 + g_6)(s) S_{0,\varepsilon}^v \delta(s) dx ds + \nu_3 \varepsilon^2 \int_0^t \|\chi'(\varepsilon x_3) S_0^v \delta(s)\|_{L^2}^2 ds.$$

By the definition of g_3, we have

$$\int_0^t \int_{\mathbb{R}^3} \chi(\varepsilon x_3) g_3(s) S_{0,\varepsilon}^v \delta(s) dx ds = \int_0^t D_1(s) ds + \int_0^t D_2(s) ds,$$

with

$$D_1 := \int_{\mathbb{R}^3} [\chi(\varepsilon x_3); S_0^v](u_1 S_0^v \delta^h) \cdot \nabla_h S_{0,\varepsilon}^v \delta dx,$$

$$D_2 := \int_{\mathbb{R}^3} S_0^v(u_1 S_{0,\varepsilon}^v \delta^h) \cdot \nabla_h S_{0,\varepsilon}^v \delta dx.$$

From Lemma 2.4, (4.5), (4.24), Hölder's inequality and Young's inequality, we obtain

$$
\begin{aligned}
\int_0^t |D_1(s)| ds &\lesssim \varepsilon^{\frac{1}{2}} \|u_1 S_0^v \delta^h\|_{L_t^2(L_h^2(L_v^\infty))} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L_t^2(L^2)} \\
&\lesssim \varepsilon^{\frac{1}{2}} \|u_1\|_{L_t^{\frac{2p}{p-1}}(L_h^{2p}(L_v^\infty))} \|S_0^v \delta^h\|_{L_t^{2p}(L_h^{\frac{2p}{p-1}}(L_v^\infty))} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L_t^2(L^2)} \\
&\leq \frac{\nu_h}{10} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L_t^2(L^2)}^2 + C\nu_h^{-1-\frac{1}{p}} \varepsilon \|u_1\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}^2 \|S_0^v \delta^h\|_{B^{0,\frac{1}{2}}(t)}^2,
\end{aligned}
$$

$$
\begin{aligned}
|D_2| &\lesssim \|u_1\|_{L_h^{2p}(L_v^\infty)} \|S_{0,\varepsilon}^v \delta\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L^2} \\
&\lesssim \|u_1\|_{L_h^{2p}(L_v^\infty)} \|S_{0,\varepsilon}^v \delta\|_{L^2}^{1-\frac{1}{p}} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L^2}^{1+\frac{1}{p}} \\
&\leq \frac{\nu_h}{10} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L^2}^2 + C\nu_h^{-\frac{p+1}{p-1}} \|u_1\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} \|S_{0,\varepsilon}^v \delta^h\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \chi(\varepsilon x_3) g_3(s) S_{0,\varepsilon}^v \delta(s) dx ds \\
&\leq \frac{\nu_h}{5} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L_t^2(L^2)}^2 + C\nu_h^{-1-\frac{1}{p}} \varepsilon \|u_1\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}^2 \|S_0^v \delta\|_{B^{0,\frac{1}{2}}(t)}^2 \\
&\quad + C\nu_h^{-\frac{p+1}{p-1}} \int_0^t \|u_1\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} \|S_{0,\varepsilon}^v \delta\|_{L^2}^2 ds. \qquad (4.25)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \chi(\varepsilon x_3) g_6(s) S_{0,\varepsilon}^v \delta(s) dx ds \\
&\leq \frac{\nu_h}{5} \|\nabla_h S_{0,\varepsilon}^v \delta\|_{L_t^2(L^2)}^2 + C\nu_h^{-1-\frac{1}{p}} \varepsilon \|u_2\|_{\widetilde{L}_t^{\frac{2p}{p-1}}(\widetilde{L}_h^{2p}(B_v^{\frac{1}{2}}))}^2 \|S_0^v \delta\|_{B^{0,\frac{1}{2}}(t)}^2 \\
&\quad + C\nu_h^{-\frac{p+1}{p-1}} \int_0^t \|u_2\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} \|S_{0,\varepsilon}^v \delta\|_{L^2}^2 ds. \qquad (4.26)
\end{aligned}
$$

From (4.21)-(4.23), (4.25)-(4.26) and Lemma 2.3, we have, $t \in [0, T_1]$,

$$\|S_{0,\varepsilon}^v \delta(t)\|_{L^2}^2 + \nu_h \int_0^t \|\nabla_h S_{0,\varepsilon}^v \delta(s)\|_{L^2}^2 ds$$

$$\leq \quad C_{\nu_h}(1+\varepsilon)C_{12}^4(T_1) + C_{\nu_h}\int_0^t \left(1 + \|u_1\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} + \|u_2\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}}\right)\|S_{0,\varepsilon}^v\delta\|_{L^2}^2 ds$$

$$+\nu_3\varepsilon^2\int_0^t \|\chi'(\varepsilon x_3)\|_{L_v^2}^2\|S_0^v\delta(s)\|_{L_h^2(L_v^\infty)}^2 ds$$

$$\leq \quad C_{\nu_h}(1+\varepsilon)C_{12}^4(T_1) + C_{\nu_h}\int_0^t \left(1 + \|u_1\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}} + \|u_2\|_{L_h^{2p}(L_v^\infty)}^{\frac{2p}{p-1}}\right)\|S_{0,\varepsilon}^v\delta\|_{L^2}^2 ds$$

$$+C\nu_3\varepsilon T_1 C_{12}^2(T_1),$$

where $C_{12}(T_1) = \|u_1\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T_1)} + \|u_2\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T_1)} + \|\delta\|_{B^{0,\frac{1}{2}}(T_1)}$. Using Gronwall's inequality and Lemma 2.3, we obtain, $t \in [0, T_1]$,

$$\|S_{0,\varepsilon}^v\delta(t)\|_{L^2}^2 + \nu_h\int_0^t \|\nabla_h S_{0,\varepsilon}^v\delta(s)\|_{L^2}^2 ds$$

$$\leq \quad \left(C_{\nu_h}(1+\varepsilon)C_{12}^4(T_1) + C\nu_3\varepsilon T_1 C_{12}^2(T_1)\right)\exp\{C_{\nu_h}(T_1 + C_{12}^{\frac{2p}{p-1}}(T_1))\}.$$

Passing to the limit when ε tends to 0 allows to conclude the proof of this lemma. $\qquad\square$

Conclusion of the proof of the uniqueness. From Lemmas 4.2-4.3, we have

$$\|\delta(t)\|_{\mathcal{H}}^2 \leq \int_0^t f(s)\mu(\|\delta(s)\|_{\mathcal{H}}^2)ds, \ t \in [0, T_1].$$

with $f(t) := C(\delta(t), u_1(t)) + C(\delta(t), u_2(t))$. Lemma 2.3 and (4.3)-(4.4) imply that $f \in L^1([0,T_1])$. Then, the uniqueness on $[0,T_1]$ follows from the Osgood Lemma (see for instance [7]). Since $u \in C([0,T]; B_p^{-1+\frac{2}{p},\frac{1}{2}})$, one can easily obtain the uniqueness of the solution u on $[0,T]$. $\qquad\square$

5 Continuous dependence

Proof of (1.5). Here, we give a sketch proof of (1.5).

From (4.5), we have

$$\|a\|_{L_h^{\frac{2p}{p-1}}(L_v^2)} \lesssim \|a\|_{L^2}^{1-\frac{1}{p}}\|\nabla_h a\|_{L^2}^{\frac{1}{p}}, \ p \geq 2. \tag{5.1}$$

Similar to (4.2), we obtain

$$\begin{cases} \delta_t - \nu_h\Delta_h\delta - \nu_3\partial_3^2\delta = -\delta \cdot \nabla u_1 - u_2 \cdot \nabla\delta - \nabla P, \\ \text{div}\delta = 0, \\ \delta|_{t=0} = \delta_0 := u_{02} - u_{01}, \end{cases} \tag{5.2}$$

where $\delta := u_2 - u_1$. By the L^2 energy estimate, (5.1) and the Cauchy-Schwarz inequality, we have, for $p \geq 2$

$$\frac{d}{dt}\|\delta\|_{L^2}^2 + 2\nu_h\|\nabla_h\delta\|_{L^2}^2 + 2\nu_3\|\partial_3\delta\|_{L^2}^2$$

$$\leq \quad 2\left|\int_{\mathbb{R}^3}(\delta \otimes u_1) : \nabla\delta dx\right| + 2\left|\int_{\mathbb{R}^3}(u_2 \otimes \delta) : \nabla\delta dx\right|$$

$$\leq \quad C\|\nabla\delta\|_{L^2}\|\delta\|_{L_h^{\frac{2p}{p-1}}(L_v^2)}\left(\|u_1\|_{L_h^{2p}(L^\infty)} + \|u_2\|_{L_h^{2p}(L^\infty)}\right)$$

$$
\begin{aligned}
&\leq\ C\|\nabla\delta\|_{L^2}\|\delta\|_{L^2}^{1-\frac{1}{p}}\|\nabla_h\delta\|_{L^2}^{\frac{1}{p}}\left(\|u_1\|_{L_h^{2p}(L^\infty)}+\|u_2\|_{L_h^{2p}(L^\infty)}\right)\\
&\leq\ C\|\nabla\delta\|_{L^2}^{1+\frac{1}{p}}\|\delta\|_{L^2}^{1-\frac{1}{p}}\left(\|u_1\|_{L_h^{2p}(L^\infty)}+\|u_2\|_{L_h^{2p}(L^\infty)}\right)\\
&\leq\ \nu_h\|\nabla_h\delta\|_{L^2}^2+\nu_3\|\partial_3\delta\|_{L^2}^2+C(\nu_h^{-\frac{p+1}{p-1}}+\nu_3^{-\frac{p+1}{p-1}})\|\delta\|_{L^2}^2\left(\sum_{i=1}^2\|u_i\|_{L_h^{2p}(L^\infty)}^{\frac{2p}{p-1}}\right).
\end{aligned}
$$

Then, we have

$$
\frac{d}{dt}\|\delta\|_{L^2}^2\leq C(\nu_h^{-\frac{p+1}{p-1}}+\nu_3^{-\frac{p+1}{p-1}})\|\delta\|_{L^2}^2\left(\sum_{i=1}^2\|u_i\|_{L_h^{2p}(L^\infty)}^{\frac{2p}{p-1}}\right),\ p\geq 2.
$$

Using Gronwall's inequality and Lemma 2.3, we obtain, for $p\geq 2$,

$$
\begin{aligned}
\|\delta\|_{L^2}^2\ &\leq\ \|\delta_0\|_{L^2}^2\exp\left\{C(\nu_h^{-\frac{p+1}{p-1}}+\nu_3^{-\frac{p+1}{p-1}})\int_0^T\left(\sum_{i=1}^2\|u_i\|_{L_h^{2p}(L^\infty)}^{\frac{2p}{p-1}}\right)ds\right\}\\
&\leq\ \|\delta_0\|_{L^2}^2\exp\left\{C\nu_h^{-1}(\nu_h^{-\frac{p+1}{p-1}}+\nu_3^{-\frac{p+1}{p-1}})\left(\sum_{i=1}^2\|u_i\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}(T)}\right)^{\frac{2p}{p-1}}\right\}.
\end{aligned}
$$

This finishes the proof of (1.5) and Theorem 1.1. $\qquad\square$

6 Proof of Proposition 1.1

From Lemmas 2.1 and 2.5, we have

$$
\|\Delta_k^h\Delta_l^v a_F\|_{L_T^q(L_h^4(L_v^2))}\lesssim\begin{cases} d_{k,l}\nu_h^{-\frac{1}{q}}2^{(\frac{1}{2}-\frac{2}{q})k}2^{-\frac{l}{2}}\|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}, & \text{for } k\geq l-1,\\[2mm] 0, & \text{else,}\end{cases}\tag{6.1}
$$

where $1\leq q\leq\infty$ and $p\in[2,4]$. Then, we obtain,

$$
\|\Delta_k^h a_F\|_{L^q(\mathbb{R}^+;L_h^4(L_v^\infty))}\lesssim\nu_h^{-\frac{1}{q}}c_k2^{-k(\frac{2}{q}-\frac{1}{2})}\|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}},\quad\text{for}\ \ q\in[1,\infty].\tag{6.2}
$$

Using Bony's decomposition in the vertical variable, we obtain

$$
\Delta_j^v(a_F^3\partial_3 a_F)=\sum_{|j-j'|\leq 5}\Delta_j^v(S_{j'-1}^v a_F^3\partial_3\Delta_{j'}^v a_F)+\sum_{j'\geq j-N_0}\Delta_j^v(\Delta_{j'}^v a_F^3\partial_3 S_{j'+2}^v a_F).
$$

The two terms of the above sum are estimated exactly along the same lines. Using Bony's decomposition in the horizontal variable, we obtain

$$
S_{j'-1}^v a_F^3\partial_3\Delta_{j'}^v a_F=\sum_{k\geq j'-N_0}\left\{S_{k-1}^h S_{j'-1}^v a_F^3\partial_3\Delta_k^h\Delta_{j'}^v a_F+\Delta_k^h S_{j'-1}^v a_F^3\partial_3 S_{k+2}^h\Delta_{j'}^v a_F\right\}.
$$

Using Hölder's inequality, Young's inequality, Lemma 2.1 and (6.1)-(6.2), we get

$$
\begin{aligned}
&\|S_{k-1}^h S_{j'-1}^v a_F^3\partial_3\Delta_k^h\Delta_{j'}^v a_F\|_{L_T^1(L^2(\mathbb{R}^3))}\\
&\leq\ \|S_{k-1}^h S_{j'-1}^v a_F^3\|_{L_T^\infty(L_h^4(L_v^\infty))}\|\partial_3\Delta_k^h\Delta_{j'}^v a_F\|_{L_T^1(L_h^4(L_v^2))}
\end{aligned}
$$

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$$\lesssim \sum_{k' \leq k-2} c_{k'} 2^{\frac{k'}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} d_{k,j'} \nu_h^{-1} 2^{\frac{j'}{2}} 2^{-\frac{3}{2}k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

$$\lesssim c_k 2^{\frac{k}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} d_{k,j'} \nu_h^{-1} 2^{\frac{j'}{2}} 2^{-\frac{3}{2}k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

$$\lesssim c_k d_{k,j'} \nu_h^{-1} 2^{\frac{j'}{2}} 2^{-k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2,$$

$$\|\Delta_k^h S_{j'-1}^v a_F^3 \partial_3 S_{k+2}^h \Delta_{j'}^v a_F\|_{L_T^1(L^2(\mathbb{R}^3))}$$

$$\leq \|\Delta_k^h S_{j'-1}^v a_F^3\|_{L_T^1(L_h^4(L_v^\infty))} \|\partial_3 S_{k+2}^h \Delta_{j'}^v a_F\|_{L_T^\infty(L_h^4(L_v^2))}$$

$$\lesssim c_k \nu_h^{-1} 2^{-\frac{3}{2}k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} \sum_{k' \leq k} d_{k',j'} 2^{\frac{j'}{2}} 2^{\frac{k'}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

$$\lesssim c_k \nu_h^{-1} 2^{-\frac{3}{2}k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}} d_{k,j'} 2^{\frac{j'}{2}} 2^{\frac{k}{2}} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}$$

$$\lesssim c_k d_{k,j'} \nu_h^{-1} 2^{\frac{j'}{2}} 2^{-k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2$$

and

$$\|S_{j'-1}^v a_F^3 \partial_3 \Delta_{j'}^v a_F\|_{L_T^1(L^2(\mathbb{R}^3))}$$

$$\lesssim \sum_{k \geq j'-N_0} c_k d_{k,j'} \nu_h^{-1} 2^{\frac{j'}{2}} 2^{-k} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2$$

$$\lesssim d_{j'} 2^{-\frac{j'}{2}} \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2.$$

Thus, we can easily obtain

$$\|\Delta_j^v(a_F^3 \partial_3 a_F)\|_{L_T^1(L^2(\mathbb{R}^3))}$$

$$\lesssim d_j 2^{-\frac{j}{2}} \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2 + \sum_{j' \geq j-N_0} d_{j'} 2^{-\frac{j'}{2}} \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2$$

$$\leq d_j 2^{-\frac{j}{2}} \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2$$

and

$$\sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \int_0^\infty \|\Delta_j^v(a_F^3 \partial_3 a_F)\|_{L^2(\mathbb{R}^3)} dt \lesssim \nu_h^{-1} \|a\|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2.$$

Using the similar argument, we can easily estimate the term $\sum 2^{\frac{j}{2}} \int_0^\infty \|\Delta_j^v(a_F^h \cdot \nabla_h a_F)\|_{L^2}$, and finish the proof of Proposition 1.1. $\qquad\square$

7 Proof of Proposition 1.2

Using methods in [2, 3], we can prove Proposition 1.2 as follows.

We shall start by estimating the high frequencies. Defining a threshold $k_0 \geq 0$ to be determined later on, we have

$$\sum_{k=k_0}^\infty 2^{-\sigma k} \|\Delta_k^h \phi_\varepsilon\|_{L^q(\mathbb{R}^2)} \leq C 2^{-\sigma k_0} \|\phi_\varepsilon\|_{L^q(\mathbb{R}^2)} = C 2^{-\sigma k_0} \|\phi\|_{L^q(\mathbb{R}^2)}$$

On the other hand, noting that $e^{i\frac{x_1}{\varepsilon}} = (-i\varepsilon\partial_1)^N(e^{i\frac{x_1}{\varepsilon}})$, we get, for any $N \in \mathbb{N}$,

$$\Delta_k^h \phi_\varepsilon = (i\varepsilon)^N 2^{2k} \sum_{l=0}^{N} \int_{\mathbb{R}^2} C_N^l e^{i\frac{y_1}{\varepsilon}} \partial_{x_1}^l (g(2^k(x_h - y_h))) \partial_{x_1}^{N-l} \phi(y_h) dy_h,$$

where $g(x_h) \in \mathcal{S}(\mathbb{R}^2)$ satisfying $\mathcal{F}g(\xi_h) = \varphi(|\xi_h|)$. Young's inequality enables us to infer that

$$\|\Delta_k^h \phi_\varepsilon\|_{L^q} \le C_\phi \varepsilon^N 2^{2k} \min\left(\sum_{l=0}^{N} 2^{k(l-2)}, \sum_{l=0}^{N} 2^{k(l-\frac{2}{q})}\right).$$

So, choosing N large enough, we obtain

$$\sum_{0 \le k \le k_0} 2^{-\sigma k} \|\Delta_k^h \phi_\varepsilon\|_{L^q} \le C_\phi \sum_{0 \le k \le k_0} 2^{k(N-\sigma)} \varepsilon^N \le C_\phi 2^{k_0(N-\sigma)} \varepsilon^N$$

and

$$\|S_0^h \phi_\varepsilon\|_{L^2(\mathbb{R}^2)} \le \sum_{k \le -1} \|\Delta_k^h \phi_\varepsilon\|_{L^2} \le \sum_{k \le -1} C_\phi \varepsilon^N 2^{2k(1-\frac{1}{2})} \le C_\phi \varepsilon^N.$$

Choosing the best k_0, we have

$$\|\phi_\varepsilon\|_{\tilde{B}_{q,1}^{-\sigma}} \le C_\phi \varepsilon^\sigma.$$

Similarly, since $\alpha < 2(1-\frac{1}{q})$, we obtain

$$\sum_{k \le -1} 2^{-\alpha k} \|\Delta_k^h \phi_\varepsilon\|_{L^q} \le \sum_{k \le -1} C_\phi \varepsilon^N 2^{k(2(1-\frac{1}{q})-\alpha)} \le C_\phi \varepsilon^N$$

and

$$\|\phi_\varepsilon\|_{\dot{B}_{q,1}^{-\alpha}} \le C_\phi \varepsilon^\alpha.$$

From (1.1) in [2] (or Definition 1.1 in [3]), we have

$$\|f\|_{\dot{B}_{q,p}^{-\sigma}(\mathbb{R}^2)} \simeq \left\| t^{\frac{\sigma}{2}} \|e^{t\Delta_h} f\|_{L^q} \right\|_{L^p(\mathbb{R}^+, \frac{dt}{t})}$$

and

$$\|\phi_\varepsilon\|_{\dot{B}_{q,\infty}^{-\sigma}(\mathbb{R}^2)} \simeq \sup_{t>0} t^{\frac{\sigma}{2}} \|e^{t\Delta_h} \phi_\varepsilon\|_{L^q} \ge C\varepsilon^\sigma \|e^{\varepsilon^2\Delta_h} \phi_\varepsilon\|_{L^q}, \quad \sigma > 0.$$

For any function g satisfying $\text{supp}\,\hat{g} \in \varepsilon^{-1}\mathcal{C}_h$, we have

$$\|\mathcal{F}^{-1}(e^{\varepsilon^2|\xi_h|^2}\hat{g})\|_{L^q} \le C\|g\|_{L^q}.$$

Since the support of $\mathcal{F}\phi_\varepsilon$ is included in $\varepsilon^{-1}\mathcal{C}_h$ for some ring $\mathcal{C}_h$, applied with $g = e^{\varepsilon^2\Delta}\phi_\varepsilon$, this inequality gives

$$\|\phi_\varepsilon\|_{L^q} \le C\|e^{\varepsilon^2\Delta}\phi_\varepsilon\|_{L^q} \quad \text{and} \quad \|\phi_\varepsilon\|_{\dot{B}_{q,\infty}^{-\sigma}} \ge C^{-1}\varepsilon^\sigma \|\phi\|_{L^q}.$$

From (1.6), we have

$$\int_0^\infty \|\phi_{\varepsilon,F'}\psi_{\varepsilon,F'}\|_{L^2(\mathbb{R}^2)} dt$$

$$\le \left(\int_0^\infty \|\phi_{\varepsilon,F'}\|_{L^4}^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty \|\psi_{\varepsilon,F'}\|_{L^4}^2 dt\right)^{\frac{1}{2}}$$

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$$\begin{aligned}
&= \nu_h^{-1} \left(\int_0^\infty \| e^{t\Delta_h} \sum_{k \geq 0} \Delta_k^h \phi_\varepsilon \|_{L^4}^2 dt \right)^{\frac{1}{2}} \left(\int_0^\infty \| e^{t\Delta_h} \sum_{k \geq 0} \Delta_k^h \psi_\varepsilon \|_{L^4}^2 dt \right)^{\frac{1}{2}} \\
&\simeq \nu_h^{-1} \| \sum_{k \geq 0} \Delta_k^h \phi_\varepsilon \|_{\dot{B}_{4,2}^{-1}} \| \sum_{k \geq 0} \Delta_k^h \psi_\varepsilon \|_{\dot{B}_{4,2}^{-1}} \\
&\leq \nu_h^{-1} \| \sum_{k \geq 0} \Delta_k^h \phi_\varepsilon \|_{\dot{B}_{4,1}^{-1}} \| \sum_{k \geq 0} \Delta_k^h \psi_\varepsilon \|_{\dot{B}_{4,1}^{-1}} \leq C_{\phi,\psi,\nu_h} \varepsilon^2.
\end{aligned}$$

This concludes the proof of Proposition 1.2. $\qquad\square$

8   An imbedding result

Proof of Proposition 1.3. It is easy to obtain that $\dot{B}_{\infty,2}^{-1} \subset BMO^{-1} \subset \dot{B}_{\infty,\infty}^{-1} = C^{-1}$, (See [2]).
Thus, we only need to prove that $B_p^{-1+\frac{2}{p},\frac{1}{2}} \subset \dot{B}_{\infty,2}^{-1}$. From Lemma 2.1 and Young's inequality, we have

$$\begin{aligned}
&\sum_{q \in \mathbb{Z}} 2^{-2q} \| \Delta_q u \|_{L^\infty}^2 \\
&= \sum_{q \in \mathbb{Z}} 2^{-2q} \| \sum_{l \in \mathbb{Z}} \Delta_q \big(\sum_{k \geq l-1} \Delta_k^h \Delta_l^v u + S_{l-1}^h \Delta_l^v u \big) \|_{L^\infty}^2 \\
&\leq \sum_{q \in \mathbb{Z}} \sum_{l \leq q+N_0} \sum_{|k-q| \leq N_0} 2^{-2q} \| \Delta_q \Delta_k^h \Delta_l^v u \|_{L^\infty}^2 + \sum_{q \in \mathbb{Z}} \sum_{l \geq q-N_0} 2^{-2q} \| \Delta_q S_{l-1}^h \Delta_l^v u \|_{L^\infty}^2 \\
&\lesssim \sum_{q \in \mathbb{Z}} \sum_{l \leq q+N_0} \sum_{|k-q| \leq N_0} 2^{-2q} 2^{2l} 2^{\frac{4k}{p}} \| \Delta_k^h \Delta_l^v u \|_{L_h^p(L_v^2)}^2 + \sum_{q \in \mathbb{Z}} \sum_{l \geq q-N_0} 2^{-2q} 2^{3q} \| S_{l-1}^h \Delta_l^v u \|_{L^2}^2 \\
&\lesssim \left(\sum_{q \in \mathbb{Z}} \sum_{l \leq q+N_0} \sum_{|k-q| \leq N_0} 2^{-2q+2k} d_{k,l}^2 + \sum_{q \in \mathbb{Z}} \sum_{l \geq q-N_0} 2^{q-l} d_l^2 \right) \| u \|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2 \\
&\lesssim \| u \|_{B_p^{-1+\frac{2}{p},\frac{1}{2}}}^2, \ p \geq 2.
\end{aligned}$$

Then, we finish the proof of Proposition 1.3. $\qquad\square$

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